# Sampling of Operators

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**Abstract**

Sampling and reconstruction of functions is a central tool in science. A key result is given by the sampling theorem for bandlimited functions attributed to Whittaker, Shannon, Nyquist, and Kotelnikov. We develop an analogous sampling theory for operators which we call bandlimited if their Kohn-Nirenberg symbols are bandlimited. We prove sampling theorems for such operators and show that they are extensions of the classical sampling theorem.[1]

## 1 Introduction

The classical sampling theorem for bandlimited functions states that a function whose Fourier transform is supported on an interval of length $\Omega$ is completely characterized by samples taken at rate at least $1/\Omega$ per unit interval. That is, with $\mathcal{F}$ denoting the Fourier transform[2] we have

**Theorem 1.1** *For $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \mathcal{F} f \subseteq [-\frac{\Omega}{2}, \frac{\Omega}{2})$, choose $T$ with $T\Omega \leq 1$. Then*

$$\|\{f(nT)\}\|_{l^2(\mathbb{Z})} = T\,\|f\|_{L^2(\mathbb{R})},$$

*and $f$ can be reconstructed by means of*

$$f(x) = \sum_{n\in\mathbb{Z}} f(nT)\,\frac{\sin(\pi T(x-n))}{\pi T(x-n)}$$

*with convergence in $L^2(\mathbb{R})$.*

Theorem 1.2 is an exemplary result from our sampling theory of operators. We choose a Hilbert–Schmidt operator $H$ on $L^2(\mathbb{R})$ with kernel $\kappa_H$ and Kohn-Nirenberg symbol $\sigma_H$, that is $\sigma_H(x, D) = H$ in the sense of pseudodifferential operators [Hör79, Tay81]. Recall that Hilbert–Schmidt operators on $L^2(\mathbb{R})$ are exactly those bounded operators $H$ with $\sigma_H \in L^2(\mathbb{R}^2)$ and corresponding norm of $H$. Let $\mathcal{F}^s$ denote the so-called symplectic Fourier transform[2] on $L^2(\mathbb{R}^{2d})$.

**Theorem 1.2** *For $H : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ Hilbert–Schmidt with $\operatorname{supp} \mathcal{F}^s\sigma_H \subseteq [0, T)\times[-\frac{\Omega}{2}, \frac{\Omega}{2})$ and $T\Omega \leq 1$, we have*

$$\|H \sum_{k\in\mathbb{Z}} \delta_{kT}\|_{L^2(\mathbb{R})} = T\|H\|_{HS},$$

[1] 2010 Mathematics Subject Classification. Primary 42B35, 94A20; Secondary 35S05, 47B35, 94A20.
[2] See Section 2 for basic notation used throughout this paper.

*and $H$ can be reconstructed by means of*

$$\kappa_H(x+t,x)=\sum_{n\in\mathbb{Z}}\big(H\sum_{k\in\mathbb{Z}}\delta_{kT}\big)(t+nT)\,\frac{\sin(\pi T(x-n))}{\pi T(x-n)}$$

*with convergence in $HS(L^2(\mathbb{R}^2)$.*

As shown in Section 3, Theorem 1.1 can be deduced from the general form of Theorem 1.2 which is stated below as Theorem 3.4.

The appearance of the sampling rate $T$ in the description of the bandlimitation of the operator's Kohn–Nirenberg symbol reflects a fundamental difference between the sampling of operators and the sampling of functions. This phenomenon is illuminated in terms of operator identification by Theorem 3.6 in [KP06] and Theorem 1.1 in [PW06a], results which we extend here by Theorems 4.6 and 4.7 below. In fact, in the classical sampling theory, the bandlimitation of a function to a large interval can be compensated by choosing a correspondingly high sampling rate. In the here developed sampling theory for operators, only bandlimitations to sets of area less than or equal to one permit sampling and reconstruction. The bandlimitation to, for example, a rectangle of area 2 cannot be compensated by increasing the sampling rate, and, in fact, operators characterized by such a bandlimitation cannot be determined in a stable manner by the application of the operator to a single function or distribution, whether it is supported on a discrete set (which we shall refer to as sampling set below), or not.

Theorems 4.6 and 4.7 in simple terms is Theorem 1.3 below. It can also be deduced from earlier operator identification results in [Pfa08a, PW06b]. As it is customary to define Paley–Wiener spaces

$$PW\big(M\big)=\{f\in L^2(\mathbb{R}^d):\,\operatorname{supp}\mathcal{F}f\subseteq M\}$$

to describe spaces of functions bandlimited to $M\subseteq\mathbb{R}^d$, we introduce in this paper operator Paley–Wiener spaces

$$OPW\big(M\big)=\{H\in HS(L^2(\mathbb{R}^d)):\,\operatorname{supp}\mathcal{F}^s\sigma_H\subseteq M\}$$

to describe operators bandlimited to $M\subseteq\mathbb{R}^{2d}$. In short, $PW(M)$ and $OPW(M)$ are linked via the Kohn-Nirenberg correspondence [Fol89, KN65].

**Theorem 1.3** *Let $\mu(M)$ denote the Lebesgue measure of the set $M\subseteq\mathbb{R}^2$.*

1. *For $M$ compact with $\mu(M)<1$ exists $T>0$, a bounded sequence $\{c_k\}$, and $A,B>0$ with*

$$A\|H\|_{HS}\leq\|H\sum_{k\in\mathbb{Z}}c_k\delta_{kT}\|_{L^2(\mathbb{R})}\leq B\|H\|_{HS},\quad H\in OPW(M).$$

2. *Let $M$ be open with $\mu(M)>1$, then, for all $g\in\mathcal{S}'(\mathbb{R})$ and $\epsilon>0$, exists $H\in OPW(M)$ with*

$$\|Hg\|_{L^2(\mathbb{R})}\leq\epsilon\|H\|_{HS}.$$

The sampling theory developed here has roots in the work of Kozek, Pfander [KP06] and Pfander, Walnut [PW06a] which addressed the identifiability of slowly time–varying operators, that is, of so–called underspread operators. Measurability or identifiability of a given operator class describes the property that all operators of that class can be distinguished by their action on a well chosen single function or distribution. The importance of operator identification and, therefore, operator sampling in engineering and science is illustrated by the following two examples. In case of information transmission,

complete knowledge of the communications channel operator at hand allows the transmitter to optimize its transmission strategy in order to transmit information close to channel capacity (see, for example, [Gol05] and references therein). In radar, simply speaking, a signal is send out and the goal is to determine the nature of reflecting objects from the received echo, that is, from the response to the radar channels input signal [Sko80, KP06].

The best known operator identification example states that time–invariant operators are fully characterized by their response to a Dirac impulse. Kailath [Kai62] and later Bello [Bel69] investigated the identifiability of slowly time varying channels (operators) which are defined by the support size of their spreading functions, namely of the symplectic Fourier transform of the operators' Kohn–Nirenberg symbols. In both papers, conjectures were made that were then proven in [KP06], respectively [PW06a]. The operator sampling Theorems 4.6 and 4.7 extend the main results in [KP06, PW06a], Shannon's sampling theorem, as well as the fact that time–invariant operators are identifiable by their impulse response (see Figure 1).

The paper is structured as follows. Section 2 provides background on time–frequency analysis of functions and distributions, in particular on modulation spaces (Section 2.1), as well as on time–frequency analysis of pseudodifferential operators (Section 2.2). In Section 2.3 we discuss boundedness of pseudodifferential operators on modulation spaces. In Sections 3 and 4 we state and prove our main results. Section 5 contains references to recent progress and open questions in the sampling theory for operators .

## 2 Background

$L^2(\mathbb{R}^d)$ denotes the Hilbert space of complex valued, Lebesgue measurable functions on Euclidean space $\mathbb{R}^d$ [Fol99]. The *Fourier transformation* $\mathcal{F}$, respectively the *symplectic Fourier transformation* $\mathcal{F}_s$, is the unitary operator $\mathcal{F}: L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),\quad f \mapsto \widehat{f} = \mathcal{F}f$, densely defined by

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \gamma\cdot x}\, dx, \quad f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d),$$

respectively $\mathcal{F}^s : L^2(\mathbb{R}^{2d}) \longrightarrow L^2(\mathbb{R}^{2d})$ with

$$\mathcal{F}^s F(t,\nu) = \iint_{\mathbb{R}^{2d}} F(x,\xi)\, e^{-2\pi i[(t,\nu),(x,\xi)]}\, dx\, d\xi = \iint_{\mathbb{R}^{2d}} F(x,\xi)\, e^{-2\pi i(\nu\cdot x-\xi\cdot t)}\, dx\, d\xi, \quad F \in L^1(\mathbb{R}^{2d})\cap L^2(\mathbb{R}^{2d}),$$

where $[\cdot,\cdot]$ denotes the symplectic form on $\mathbb{R}^{2d}$. Throughout the paper, integration is with respect to the Lebesgue measure which we denote by $\mu$.

The Fourier transform defines isomorphisms on the Frechet space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ and on its dual $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions (equipped with the weak-$*$ topology). Note that $\mathcal{S}'(\mathbb{R}^d)$ contains constant functions, *Dirac's delta* $\delta : f \mapsto f(0)$, and weighted *Shah distributions* $\sum_{n\in\mathbb{Z}^d} c_n \delta_{kT}$, $T \in (\mathbb{R}^+)^d$, with $\{c_n\}$ having at most polynomial growth.

Similarly to the *Fourier transformation*, the *time shift operator* $T_t$, $t \in \mathbb{R}^d$, given by $T_t f(x) = f(x-t)$ and the *modulation operator* $M_w$, $w \in \mathbb{R}^d$, $M_w f(x) = e^{2\pi i w\cdot x} f(x)$, act as unitary operators on $L^2(\mathbb{R})$ and they are isomorphisms on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. Note that $M_w$ is also called *frequency shift operator* since $\widehat{M_w f} = T_w \widehat{f}$. Further, we refer to $\pi(\lambda) = \pi(t,\nu) = M_\nu T_t$ for $\lambda = (t,\nu) \in \mathbb{R}^{2d}$ as *time–frequency shift operator*. Note that we have $\mathcal{F}\circ\pi(t,\nu) = e^{2\pi i t\nu}\pi(\nu,-t)\circ\mathcal{F}$, that is, $\mathcal{F}\pi(t,\nu)f = e^{2\pi i t\nu}\pi(\nu,-t)\widehat{f}$ for $f \in \mathcal{S}'(\mathbb{R}^d)$.

The goal of *operator identification* is to select, for given spaces $X$ and $Y$ of functions or distributions defined on $\mathbb{R}^d$ and a given space of linear operators $\mathcal{H}$ mapping $X$ to $Y$, an element $g \in X$ which induces a continuous, open, and injective map $\Phi_g : \mathcal{H} \longrightarrow Y(\mathbb{R}^d),\ H \mapsto Hg$ (see Figure 2).

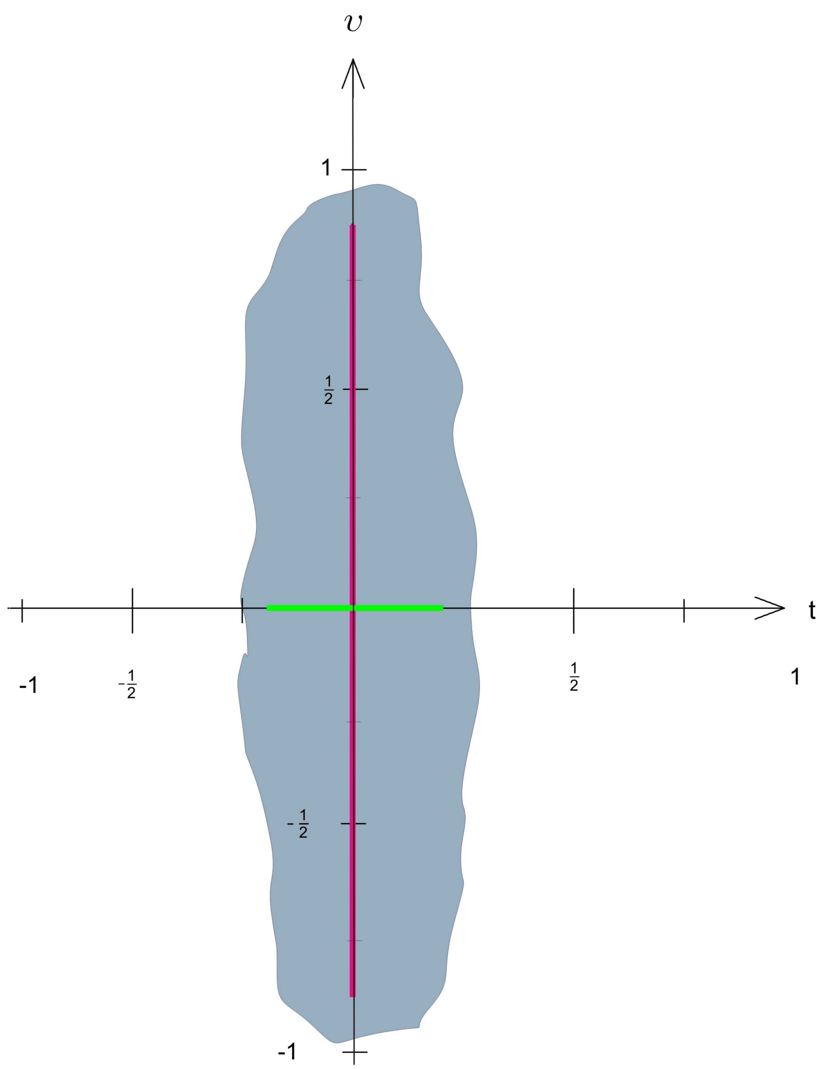


Figure 1: In the one dimensional case, the herein developed sampling theory for operators applies to any pseudodifferential operators whose Kohn–Nirenberg symbol is bandlimited to a compact set of Lebesgue measure less than one (for example, the blue region above). The results extend the classical sampling theorem described in Theorem 1.1 which is equivalent to the identifiability of operators whose Kohn-Nirenberg symbol is bandlimited to a segment of the frequency shift axis (red). Also, the fact that time–invariant operators with compactly supported impulse response can be identified from their action on the Dirac impulse is a special case of our results since the Kohn–Nirenberg symbols of time–invariant operators are bandlimited to the time shift axis (green).

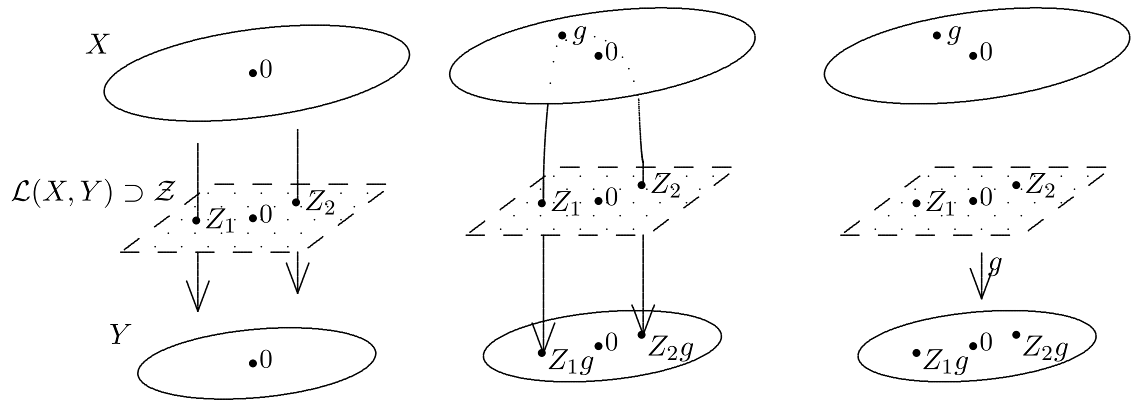


Figure 2: Illustration of the operator identification and sampling problem. We seek an element $g \in X$ in the domain of the operator class $\mathcal{Z}$ which induces a map from $\mathcal{Z}$ into the range space $Y$ which is continuous, open, and injective. If we can choose $g = \sum_j c_j \delta_{x_j}$, then $\mathcal{Z}$ permits operator sampling.

**Definition 2.1** *Let $X$ be a set, $Y$ a topological vector space, and $\mathcal{Z}$ a topological vector space of operators mapping $X$ to $Y$. The space $\mathcal{Z}$ is* identifiable by $g \in X$ *if $\Phi_g : \mathcal{Z} \longrightarrow Y$, $H \mapsto Hg$ is continuous, open and injective. In the case that $Y$ and $\mathcal{H}$ are normed spaces, this reads: there exist $A, B > 0$ with*

$$A \, \|H\|_{\mathcal{Z}} \leq \|Hg\|_Y \leq B \, \|H\|_{\mathcal{Z}}, \quad H \in \mathcal{Z}\,. \tag{1}$$

*If we can choose $g \in X = X(\mathbb{R}^d)$ of the form $g = \sum_j c_j \delta_{x_j}$, $x_j \in \mathbb{R}^d$ and $c_j \in \mathbb{C}$ for $j \in \mathbb{Z}^d$, as identifier, then we say that $\mathcal{Z}$ mapping $X$ to $Y$ permits operator sampling and we call $\{x_j\}$ a set of sampling for $\mathcal{Z}$ with respective sampling weights $\{c_j\}$. We refer to $g$ as a sampling function for the operator class $\mathcal{Z}$.*

In the following, we shall abbreviate norm equivalences as the one given in (1) using the symbol $\asymp$. For example, (1) becomes

$$\|H\|_{\mathcal{Z}} \asymp \|Hg\|_Y\,, \quad H \in \mathcal{Z}\,.$$

We shall describe in Section 2.1 the distribution spaces and in Section 2.2 the pseudodifferential operator spaces considered here. Section 2.3 discusses boundedness of the considered pseudodifferential operators on modulation spaces.

## 2.1 Modulation spaces

To describe the full scope of operator sampling, we need to employ recent results in time–frequency analysis, in particular, we have to enter the realm of so-called modulation spaces. As Theorems 1.2 and 1.3 indicate, all results presented include the special case of Hilbert–Schmidt operators and the Hilbert space of square integrable functions as range space, and we advise readers without significant expertise in time–frequency analysis to focus on this case during a first reading.

Feichtinger introduced modulation spaces in [Fei81]. Modulation space theory was further developed by Feichtinger and Gröchenig as special case of their coorbit theory [FG89]: for $\rho$ being a square

integrable unitary and irreducible representation of a locally compact group $G$ on a Hilbert space $H$ and $Y$ being a Banach space of functions on $G$, we consider, for appropriate $\varphi \in H$, the so–called *voice transform* $V_\varphi : H \longrightarrow Y$ given by $V_\varphi f(x) = \langle f, \rho(x)\varphi\rangle$, $x \in G$. Given an appropriate Banach space Gelfand triple $X \subseteq H \subseteq X'$, the *coorbit space* $M_Y$ consists of those $f \in X'$ with $\|f\|_{M_Y} = \|V_\varphi f\|_Y < \infty$ [FK98].

The special case of modulation spaces is based on the Schrödinger representation of the reduced Weyl–Heisenberg group. The corresponding voice transform simplifies to the short–time Fourier transform, that is, for any Schwartz class function $\varphi \neq 0$ we consider

$$V_\varphi f(\lambda) = \langle f, \pi(\lambda)\varphi\rangle = \mathcal{F}(f\, T_t\overline{\varphi})(\nu), \quad \lambda = (t,\nu) \in \mathbb{R}^{2d},$$

which is well defined for any $f \in \mathcal{S}'(\mathbb{R}^d)$ [Grö01]. Note that throughout this paper, dual pairings $\langle \cdot, \cdot \rangle$ are linear in the first component and antilinear in the second. Moreover, any choice of $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ can be used to define modulations spaces (with equivalent norms), but as is customary, we shall choose a normalized Gaussian, namely $\varphi(x) = \mathfrak{g}(x) = 2^{\frac{d}{4}}\, e^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$.

The role of the Banach space $Y$ in coorbit space theory is attained in modulation space theory by weighted mixed $L^p$ spaces which we shall describe now. For a measurable function $f$ on $\mathbb{R}^d$ and $p = (p_1, \dots, p_d)$, $1 \leq p_1, \dots, p_d \leq \infty$, we define mixed $L^p(\mathbb{R}^d)$ spaces by finiteness of

$$\|f\|_{L^p} = \Big( \int \Big( \dots \Big( \int \Big( \int |f(x_1, \dots, x_d)|^{p_1} dx_1 \Big)^{p_2/p_1} dx_2 \Big)^{p_3/p_2} \dots dx_{d-1} \Big)^{p_d/p_{d-1}} dx_d \Big)^{1/p_d},$$

with the usual adjustments if some $p_k = \infty$ [BP61]. The mixed $l^p(\mathbb{Z}^d)$ spaces are defined accordingly.

Note the sensitivity to the order of exponentiation and integration. For example, for $f(x,y) = 1$ if $|x-y| \leq 1$ and $f(x,y) = 0$ else, we have $\sup_x \int |f(x,y)|\, dy = 2$ but $\int \sup_x |f(x,y)|\, dy = \infty$, that is, $f \in L^{1,\infty}(\mathbb{R}^2)$ but $g \notin L^{\infty,1}(\mathbb{R}^2)$ where $g(x,y) = f(y,x)$.

A locally integrable function $v : \mathbb{R}^d \longrightarrow \mathbb{R}_0^+$ with

$$v(x+y) \leq v(x)\, v(y), \quad x, y \in \mathbb{R}^d,$$

is called *submultiplicative weight*. For example, $w_s(x) = (1 + \|x\|)^s$, $s \geq 0$, is a submultiplicative weight on $\mathbb{R}^d$. If $v$ is a submultiplicative weight and the locally integrable function $w : \mathbb{R}^d \longrightarrow \mathbb{R}_0^+$ satisfies for some $C > 0$

$$w(x+y) \leq C\, w(x)\, v(y), \quad x, y \in \mathbb{R}^d,$$

then $w$ is a *$v$-moderate weight function*. The class of $v$-moderate weight functions on $\mathbb{R}^d$ is denoted by $\mathcal{M}_v(\mathbb{R}^d)$. Note that for $s < 0$, for example, $1 \otimes w_s(x,\xi) = (1 + \|\xi\|)^s$ is not submultiplicative, but $1 \otimes w_s$ is $1 \otimes w_{-s}$ -moderate. If $w$ is a $v$-moderate weight function with respect to some submultiplicative weight, then we simply say $w$ is *moderate*. Note that for any moderate weight function on $\mathbb{R}^d$ exists $\gamma, C > 0$ with $\frac{1}{C} e^{-\gamma\|x\|_\infty} \leq w(x) \leq C e^{\gamma\|x\|_\infty}$ (see Lemma 4.2 in [Grö07]). A moderate weight function $w$ on $\mathbb{R}^d$ is a *subexponential weight function* if there exists $\gamma, C > 0$ and $0 < \beta < 1$ with

$$\tfrac{1}{C} e^{-\gamma\|x\|_\infty^\beta} \leq w(x) \leq C e^{\gamma\|x\|_\infty^\beta}.$$

Weight functions on discrete groups such as $\mathbb{Z}^d$ are defined accordingly. See [Grö07] for a thorough discussion on the role of weight functions in time–frequency analysis.

Given a $v$-moderate weight function $w$, then the Banach space $L^p_w(\mathbb{R}^d)$ is defined through finiteness of the norm $\|f\|_{L^p_w} = \|wf\|_{L^p}$. The space $L^p_w(\mathbb{R}^d)$ is shift invariant, shift operators are bounded on $L^p_w(\mathbb{R}^d)$ but not isometric if $w$ is not constant. Replacing $\mathbb{R}^d$ with $\mathbb{Z}^d$, or with a full rank lattice $\Lambda = A\mathbb{Z}^d$, $A \in \mathbb{R}^{d\times d}$ invertible, either equipped with the counting measure gives the definition of $l^p_w(\mathbb{Z}^d)$, respectively $l^p_w(\Lambda)$. If $w$ is a moderate weight on $\mathbb{R}^d$, then its restriction to $\Lambda$, which we denote by $\widetilde{w}$, is moderate as well.

**Definition 2.2** *For $p=(p_1,\dots,p_d)$ and $q=(q_1,\dots,q_d)$, $1\le p_k,q_k\le\infty$, and $w$ moderate on $\mathbb{R}^{2d}$, we define modulation spaces by*

$$M^{p,q}_w(\mathbb{R}^d)=\Big\{f\in\mathcal{S}'(\mathbb{R}^d):\ V_{\mathfrak{g}}f\in L^{p,q}_w(\mathbb{R}^{2d})\Big\} \tag{2}$$

*[Fei81, Grö01]. The modulation space $M^{p,q}_w(\mathbb{R}^d)$ is a shift invariant Banach space with norm $\|f\|_{M^{p,q}_w}=\|wV_{\mathfrak{g}}f\|_{L^{p,q}}$. If $w\equiv 1$, then we write $M^{p,q}(\mathbb{R}^d)=M^{p,q}_w(\mathbb{R}^d)$. If $p_1=\ldots=p_d$ and $q_1=\ldots=q_d$ then we abbreviate $M^{p_1,q_1}_w(\mathbb{R}^d)=M^{(p_1,\dots,p_d),(q_1,\dots,q_d)}_w(\mathbb{R}^d)$.*

Below we shall use the fact that replacing $\mathfrak{g}$ with any other $\varphi\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ in (2) defines the identical space with an equivalent norm [Grö01]. Note that if $p_1\le p_2$, $q_1\le q_2$, and $w_1\ge cw_2$ for some $c>0$, then $M^{p_1,q_1}_{w_1}$ embeds continuously in $M^{p_2,q_2}_{w_2}$, and, consequently, if $w_1\asymp w_2$ then $M^{p,q}_{w_2}(\mathbb{R}^d)=M^{p,q}_{w_1}(\mathbb{R}^d)$ with equivalent norms.

The space $M^{1,1}(\mathbb{R}^d)$ is the Feichtinger algebra, often also denoted by $S_0(\mathbb{R}^d)$, and $M^{\infty,\infty}(\mathbb{R}^d)$ is its dual $S_0'(\mathbb{R}^d)$. In fact, in general we have $M^{p,q}_w(\mathbb{R}^d)'=M^{p',q'}_{1/w}(\mathbb{R}^d)$ for $1\le p,q<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$. Note that $M^{2,2}_{1\otimes w_s}(\mathbb{R}^d)$ is also known as Bessel potential spaces, in particular $L^2(\mathbb{R}^d)=M^{2,2}(\mathbb{R}^d)$.

To illustrate the chosen order of exponentiation and integration in the definition of the modulation space $M^{p,q}_w(\mathbb{R}^d)$ for $d>1$ and $p\neq q$, we state exemplary that $f\in M^{(2,3),(4,5)}_{1\otimes w_s}(\mathbb{R}^d)$ if and only if

$$\int\Big(\int\Big(\int\Big(\int\Big|(1+\sqrt{\nu_1^2+\nu_2^2})^s\,V_{\mathfrak{g}}f(t_1,t_2,\nu_1,\nu_2)\Big|^2dt_1\Big)^{\frac{3}{2}}dt_2\Big)^{\frac{4}{3}}d\nu_1\Big)^{\frac{5}{4}}d\nu_2\le\infty.$$

Clearly $f\otimes g\in M^{(p_1,p_2),(q_1,q_2)}_{w_1\otimes w_2}(\mathbb{R}^{2d})$ if and only if $f\in M^{p_1,q_1}_{w_1}(\mathbb{R}^d)$ and $g\in M^{p_2,q_2}_{w_2}(\mathbb{R}^d)$. In this case, $\|f\otimes g\|_{M^{(p_1,p_2),(q_1,q_2)}_{w_1\otimes w_2}}=\|f\|_{M^{p_1,q_1}_{w_1}}\|g\|_{M^{p_2,q_2}_{w_2}}$.

For compactly supported and bandlimited functions, modulation spaces reduce to weighted mixed $L^p(\mathbb{R}^d)$ spaces. The following is a straight forward generalization of results in [Fei89, Oko09].

**Lemma 2.3** *Let $p=(p_1,\dots,p_d)$ and $q=(q_1,\dots,q_d)$ with $1\le p_k,q_k\le\infty$, let $w=w_1\otimes w_2$ be a moderate weight function on $\mathbb{R}^{2d}$, and suppose $M\subseteq\mathbb{R}^d$ compact. Then*

1. $\|f\|_{M^{p,q}_w}\asymp\|\widehat{f}\|_{L^q_{w_2}},\quad f\in\mathcal{S}'(\mathbb{R}^d),\ \operatorname{supp}f\subseteq M;$
2. $\|f\|_{M^{p,q}_w}\asymp\|f\|_{L^p_{w_1}},\quad f\in\mathcal{S}'(\mathbb{R}^d),\ \operatorname{supp}\widehat{f}\subseteq M.$

Modulation spaces allow for descriptions based on growth conditions of so-called Gabor coefficients [Grö01]. These descriptions rely on the following terminology.

**Definition 2.4** *Let $X$ be a Banach space, $1\le p_1,\dots,p_d\le\infty$, and let $w$ be moderate on the full rank lattice $\Lambda$.*

1. *$\{g_\lambda\}_{\lambda\in\Lambda}\subseteq X'$ is called $l^p_w$–frame for $X$, if the analysis operator $C_{\{g_\lambda\}}:X\longrightarrow l^p_w(\Lambda)$, $\quad f\mapsto\{\langle f,g_\lambda\rangle\}_{\lambda\in\Lambda}$, is well defined and*
$$\|f\|_X\asymp\|\{\langle f,g_\lambda\rangle\}\|_{l^p_w},\quad f\in X. \tag{3}$$
2. *$\{g_\lambda\}_{\lambda\in\Lambda}\subseteq X$ is called $l^p_w$–Riesz basis in $X$, if the synthesis operator $D_{\{g_\lambda\}}:l^p_w(\Lambda)\longrightarrow X$, $\quad\{c_\lambda\}_{\lambda\in\Lambda}\mapsto\sum_\lambda c_\lambda g_\lambda$, is well defined and*
$$\|\{c_\lambda\}\|_{l^p_w}\asymp\|\sum_\lambda c_\lambda g_\lambda\|_X,\quad\{c_\lambda\}\in l^p_w(\Lambda). \tag{4}$$

In the classical Hilbert space setting $X = X' = H$ and $l^p_w(\mathbb{Z}^{2d}) = l^2(\mathbb{Z}^{2d})$, the above entails the definition of Hilbert space frames and Riesz basis sequences. In the Hilbert space theory, condition (3) implies that $C_{\{g_\lambda\}}$ has a bounded left inverse, but in the general Banach space setting, (3) alone does not guarantee the existence of such a left inverse. Therefore, the condition of a bounded left inverse $C_{\mathcal{F}}$ is frequently included in the definition of frames for Banach spaces [Chr03, Grö91, FZ98].

Note that for any $1 \leq p \leq \infty$, $w$ moderate, $l^p_w$–Riesz bases form unconditional bases for their closed linear span. This follows directly from (4) and Definition 12.3.1 and Lemma 12.3.6 in [Grö01].

For $g \in \mathcal{S}(\mathbb{R}^d)$ and $\Lambda$ being a full rank lattice in $\mathbb{R}^{2d}$, we set $(g, \Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda}$. For $w$ moderate on $\mathbb{R}^{2d}$, set $\widetilde{w}_\lambda = w(\lambda)$. Results as Theorem 2.5 are important tools in modulation space theory, see, for example, Theorem 20 in [Grö04] or Theorem 6.11 in [Grö07].

**Theorem 2.5** *Let $1 \leq p, q \leq \infty$ and let $w$ be moderate on $\mathbb{R}^{2d}$. Let $\Lambda$ be a full rank lattice in $\mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d)$.*

1. *If $(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then $(g, \Lambda)$ is an $l^{p,q}_{\widetilde{w}}$–frame for $M^{p,q}_w(\mathbb{R}^d)$.*
2. *If $(g, \Lambda)$ is a Riesz basis in $L^2(\mathbb{R}^d)$, then $(g, \Lambda)$ is an $l^{p,q}_{\widetilde{w}}$–Riesz basis in $M^{p,q}_w(\mathbb{R}^d)$.*

*Proof. 1.* Let $g \in \mathcal{S}(\mathbb{R}^d)$ with $(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. Let $\widetilde{g}$ generate the canonical dual frame of $(\widetilde{g}, \Lambda)$ of $(g, \Lambda)$ [Grö01]. We have $\widetilde{g} \in \mathcal{S}(\mathbb{R}^d)$ [Jan95] and conclude that both, $C_{(g,\Lambda)} : M^{p,q}_w(\mathbb{R}^d) \longrightarrow l^{p,q}_{\widetilde{w}}(\Lambda)$ and $D_{(\widetilde{g},\Lambda)} : l^{p,q}_{\widetilde{w}}(\Lambda) \longrightarrow M^{p,q}_w(\mathbb{R}^d)$ are bounded operators. As $D_{(\widetilde{g},\Lambda)} \circ C_{(g,\Lambda)}$ is the identity on $L^2(\mathbb{R}^d)$, we can use a density argument to obtain $D_{(\widetilde{g},\Lambda)} \circ C_{(g,\Lambda)}$ is the identity on $M^{p,q}_w(\mathbb{R}^d)$. We conclude that $C_{(g,\Lambda)}$ is bounded below.

The proof of *2.* follows similarly. □

## 2.2 Time–frequency analysis of pseudodifferential operators

The framework of Hilbert–Schmidt operators suffices to develop the basics of our sampling theory for operators. But important operators such as convolution operators, multiplication operators, and even the identity are not compact and thereby fall outside the realm of Hilbert–Schmidt operators. Rather than focusing on operators with kernel in $L^2(\mathbb{R}^{2d})$, we shall consider kernels and symbols in modulation spaces.

To formulate a widely applicable sampling theory for operators, we use the general correspondence of operators to distributional kernels given by the Schwartz kernel theorem (see, for example, [Hör07]).

**Theorem 2.6** *For any linear and continuous operator $H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ exists a unique $\kappa_H \in \mathcal{S}'(\mathbb{R}^{2d})$ with $\langle Hf, g\rangle = \langle \kappa_H, \overline{f} \otimes g\rangle$, $f, g \in \mathcal{S}(\mathbb{R}^d)$.*

Alternatively to $\kappa_H$, we can consider the so–called time–varying impulse response $h_H \in S'(\mathbb{R}^{2d})$ of $H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ which is formally given by

$$h_H(x,t) = \kappa_H(x, x-t), \quad Hf(x) = \int h_H(x,t) f(x-t)\, dt.$$

The Kohn–Nirenberg symbol $\sigma_H$ of an operator $H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ is densely defined by $\sigma_H = \mathcal{F}_{t\to\xi} h_H$, that is,

$$\sigma_H(x,\xi) = \int \kappa_H(x, x-t)\, e^{-2\pi i t\xi}\, dt, \quad Hf(x) = \int \sigma_H(x,\xi)\, \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi$$

[Fol89, KN65]. Note that the $n$-th order linear differential operator $D : f \mapsto \sum_{n=0}^{N} a_n(x) f^{(n)}(x)$ has Kohn–Nirenberg symbol $\sigma_D(x,\xi) = \sum_{n=0}^{N} a_n(x)(2\pi i\xi)^n$ which is polynomial in $\xi$. Pseudodifferential operator classes, for example, those considered by Hörmander, have symbols $\sigma_H$ which are not necessarily polynomial in $\xi$ but which satisfy corresponding polynomial growth conditions [Hör07].

Additionally, in time–frequency analysis and in communications engineering, the spreading function $\eta_H$ is commonly used to describe $H$:

$$\eta_H = \mathcal{F}^s \sigma_H, \quad Hf(x) = \iint \eta_H(t,\nu)\, M_\nu T_t f(x)\, dt\, d\nu\,. \tag{5}$$

Equation (5) can be validated weakly by first integrating with respect to $x$ in

$$\langle Hf, \varphi\rangle \;=\; \int\iint \eta_H(t,\nu)\pi(t,\nu)f(x)\overline{\varphi(x)}\, dt\, d\nu\; dx = \langle \eta_H, V_f\varphi\rangle, \quad f,\varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $V_f\varphi(t,\nu) = \langle \varphi, \pi(t,\nu)f\rangle$ is the short time Fourier transform defined above. Equation (5) illustrates that support restrictions on $\eta_H$ reflect limitations on the maximal time and frequency shifts which the input signals undergo: $Hf$ is a continuous superposition of time–frequency shifted versions of $f$ with weight function $\eta_H$ [GP08, KP06, PW06a]. Moreover, as $h(x,t) = \int \eta(t,\nu)e^{2\pi i\nu x}\, d\nu$, the condition $\operatorname{supp}\eta_H(t,\cdot) \subseteq [-\frac{b}{2}, \frac{b}{2}]$, $t \in \mathbb{R}$, excludes high frequencies and therefore rapid change of the time–varying impulse response $h(x,t)$ as a function of $x$. In the time–invariant case, $\kappa(x, x-t) = h(x,t) = h(t)$ is, in fact, independent of $x$. These observations illuminate the role of support constraints on spreading functions in the analysis of *slowly time–varying* communications channels [Bel64, Zad52]. Additional aspects on the use of pseudodifferential operator calculus in communications can be found in [Str06].

### 2.3 Boundedness of pseudodifferential operators on modulation spaces

Theorem 3.3 in Section 3 provides the upper bound in (1) for Theorems 1.3, 3.4, 3.6, 4.2, and 4.6. It follows from Theorem 2.7 which generalizes Theorem 4.2 in [Tof07] as well as results in [CG03, Cza03, GH99, GH04, Tac94, Tof04] where, generally, the case $p_3 = q_3$ and $p_4 = q_4$ in the notation below was considered. Recall that $p'$ denotes the conjugate exponent of $1 \leq p \leq \infty$, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

**Theorem 2.7** *Assume* $1 \leq p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \leq \infty$ *with* $p_4 \leq q_3, q_4$,

$$1 + \frac{1}{p_2} \leq \frac{1}{p_1} + \frac{1}{p_3} + \frac{1}{p_4} \quad \textit{and} \quad 1 + \frac{1}{q_2} \leq \frac{1}{q_1} + \frac{1}{q_3} + \frac{1}{q_4}\,. \tag{6}$$

*Let the moderate weight functions* $w, w_1, w_2$ *satisfy*

$$w(x,\xi,\nu,t) \geq c\, \frac{w_2(x, \nu+\xi)}{w_1(t-x, \xi)}$$

*with* $c > 0$. *Then, for some* $C > 0$,

$$\|L_\sigma f\|_{M^{p_2,q_2}_{w_2}} \leq C\, \|\sigma\|_{M^{(p_3,q_3),(q_4,p_4)}_{w}}\, \|f\|_{M^{p_1,q_1}_{w_1}}, \quad f \in M^{p_1,q_1}_{w_1}(\mathbb{R}^d),\ \sigma \in M^{(p_3,q_3),(q_4,p_4)}_{w}(\mathbb{R}^{2d}), \tag{7}$$

*consequently,* $L_\sigma : M^{p_1,q_1}_{w_1}(\mathbb{R}^d) \longrightarrow M^{p_2,q_2}_{w_2}(\mathbb{R}^d)$ *is bounded for* $\sigma \in M^{(p_3,q_3),(q_4,p_4)}_{w}(\mathbb{R}^{2d})$ *and* $L_\sigma$ *denoting the operator corresponding to the Kohn-Nirenberg symbol* $\sigma$.

Theorem 2.7 is a consequence of the following lemma.

**Lemma 2.8** *Assume* $1 \le p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \le \infty$ *with* $p_3 \le p_1, p_2, p_4,\ \ q_3 \le q_1, q_2, q_4$,

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \quad \textit{and} \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}. \tag{8}$$

*Let the moderate weight functions* $w, w_1, w_2$ *satisfy*

$$w(x,t,\nu,\xi) \le w_1(t-x,\xi) w_2(x,\nu+\xi)\,. \tag{9}$$

*Then for* $\mathfrak{G}(x,\xi) = \mathfrak{g}(x)\mathfrak{g}(x-t)$, *we have*

$$\begin{aligned}\Big(\int\Big(\int\Big(\int\Big(\int \left|V_{\mathfrak{G}}\overline{f}\otimes g\circ\left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)(x,t,\nu,\xi)\, w(x,t,\nu,\xi)\right|^{p_3} dx\Big)^{\frac{p_4}{p_3}} dt\Big)^{\frac{q_3}{p_4}} d\xi\Big)^{\frac{q_4}{q_3}} d\nu\Big)^{\frac{1}{q_4}} \\ \le \ \|f\|_{M^{p_1,q_1}_{w_1}}\ \|g\|_{M^{p_2,q_2}_{w_2}}, \quad f \in M^{p_1,q_1}_{w_1}(\mathbb{R}^d), g \in M^{p_2,q_2}_{w_2}(\mathbb{R}^d)\,,\end{aligned} \tag{10}$$

*where* $\overline{f}\otimes g\circ\left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)(x,t) = \overline{f}(x-t)g(x)$.

*Proof.* For $g, f \in \mathcal{S}(\mathbb{R}^d)$, we compute

$$\begin{aligned} V_{\mathfrak{G}}\overline{f}\otimes g\circ\left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)(x,t,\nu,\xi) &= \iint g(x')\,\overline{f}(x'-t')e^{-2\pi i(x'\nu+t'\xi)}\mathfrak{g}(x'-x)\mathfrak{g}(x'-x-(t'-t))\,dt'\,dx' \\ &= \int g(x')e^{-2\pi i x'\nu}\mathfrak{g}(x'-x)\int \overline{f}(s)e^{-2\pi i(x'-s)\xi}\mathfrak{g}(s-(x-t))\,ds\,dx' \\ &= \int g(x')e^{-2\pi i x'(\nu+\xi)}\mathfrak{g}(x'-x)\,dx'\ \overline{\int f(s)e^{-2\pi i s\xi}\mathfrak{g}(s-(x-t))\,ds} \\ &= V_{\mathfrak{g}}g(x,\nu+\xi)\ \overline{V_{\mathfrak{g}}f(x-t,\xi)}\,. \end{aligned} \tag{11}$$

Assume $1 \le p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 < \infty$. For $w \equiv 1$ and $w_1 = w_2 \equiv 1$ we have

$$\begin{aligned} &\Big(\int\Big(\int\Big(\int\Big(\int \left|V_{\mathfrak{G}}\overline{f}\otimes g\circ\left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)(x,t,\nu,\xi)\right|^{p_3} dx\Big)^{\frac{p_4}{p_3}} dt\Big)^{\frac{q_3}{p_4}} d\xi\Big)^{\frac{q_4}{q_3}} d\nu\Big)^{\frac{1}{q_4}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |V_{\mathfrak{g}}f(x-t,\xi)\ V_{\mathfrak{g}}g(x,\nu+\xi)|^{p_3}\,dx\Big)^{\frac{p_4}{p_3}} dt\Big)^{\frac{q_3}{p_4}} d\xi\Big)^{\frac{q_4}{q_3}} d\nu\Big)^{\frac{1}{q_4}} \qquad (12) \\ &\le \Big(\int\Big(\int \|V_{\mathfrak{g}}f(\cdot,\xi)^{p_3}\|_{L^{r_1}}\|V_{\mathfrak{g}}g(\cdot,\nu+\xi)^{p_3}\|_{L^{s_1}}\Big)^{\frac{p_4}{p_3}\frac{q_3}{p_4}} d\xi\Big)^{\frac{q_4}{q_3}} d\nu\Big)^{\frac{1}{q_4}} \qquad (13) \\ &= \Big(\int\Big(\int \|V_{\mathfrak{g}}f(\cdot,\xi)^{p_3}\|_{L^{r_1}}\|V_{\mathfrak{g}}g(\cdot,\nu+\xi)^{p_3}\|_{L^{s_1}}\Big)^{\frac{q_3}{p_3}} d\xi\Big)^{\frac{q_4}{q_3}} d\nu\Big)^{\frac{q_3}{q_4}\frac{q_4}{q_3}\frac{1}{q_4}} \\ &\le \Big\|\ \|V_{\mathfrak{g}}f^{p_3}\|_{L^{r_1}}^{\frac{q_3}{p_3}}\Big\|_{L^{r_2}}^{\frac{1}{q_3}}\ \ \Big\|\ \|V_{\mathfrak{g}}g^{p_3}\|_{L^{s_1}}^{\frac{q_3}{p_3}}\Big\|_{L^{s_2}}^{\frac{1}{q_3}} \qquad (14) \\ &= \Big(\int\Big(\int |V_{\mathfrak{g}}f|^{p_3 r_1}\,dx\Big)^{\frac{r_2}{r_1}\frac{q_3}{p_3}} d\xi\Big)^{\frac{1}{r_2 q_3}}\ \ \Big(\int\Big(\int |V_{\mathfrak{g}}g^{p_3 s_1}\,dx\Big)^{\frac{s_2}{s_1}\frac{q_3}{p_3}} d\xi\Big)^{\frac{1}{s_2 q_3}} \\ &\equiv \|f\|_{M^{p_3 r_1, q_3 r_2}}\ \ \|g\|_{M^{p_3 s_1, q_3 s_2}}\,. \end{aligned}$$

To apply Young's inequality to obtain (13), we assume $p_4 \ge p_3$ and choose $r_1, s_1 \ge 1$ with

$$\frac{1}{r_1} + \frac{1}{s_1} = 1 + \frac{p_3}{p_4}. \tag{15}$$

Similarly, to obtain (14), we use $q_4 \ge q_3$ and choose $r_2, s_2 \ge 1$ with

$$\frac{1}{r_2} + \frac{1}{s_2} = 1 + \frac{q_3}{q_4}. \tag{16}$$

To conclude our proof of the unweighted case, we set $p_1 = p_3 r_1$, $q_1 = q_3 r_2$, $p_2 = p_3 s_1$, and $q_2 = q_3 s_2$. As all factors must be greater than or equal to one, we require $p_1, p_2 \geq p_3$ and $q_1, q_2 \geq q_3$. Moreover, (15) and (16) need to be satisfied, this holds if and only if (8) holds. The case that for some $k$, $p_k = \infty$ or $q_k = \infty$ follows from making the usual adjustments.

The weighted case follows by simply replacing $V_{\mathfrak{G}} G$ with $w\, V_{\mathfrak{G}} G$ in equations (12) till (13), and then replacing $V_{\mathfrak{g}} f$ and $V_{\mathfrak{g}} g$ by $w_1\, V_{\mathfrak{g}} f$ and $w_2 V_{\mathfrak{g}} g$. This is justified by (9). $\square$

*Proof of Theorem 2.7.* Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $H$ with $\sigma_H \in M^{(p_3,q_3),(q_4,p_4)}(\mathbb{R}^{2d})$. Then

$$\begin{aligned} \big|\langle Hf, g\rangle\big| &= \Big|\int\!\!\int h(x,t) f(x-t)\, dt\, \overline{g}(x)\, dx\Big| = \big|\langle h_H, \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\rangle\big| \\ &= \big|\langle \sigma_H, \mathcal{F}_{t\to\xi} \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\rangle\big| \\ &\leq \big\|\sigma_H\big\|_{M_w^{(p_3,q_3),(q_4,p_4)}} \big\|\mathcal{F}_{t\to\xi} \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big\|_{M_{1/w}^{(p_3',q_3'),(q_4',p_4')}}, \end{aligned} \tag{17}$$

where we applied Hölder's inequality for weighted mixed $L^p$-spaces to obtain (17) [Grö01]. To obtain (7), it suffices to show $\mathcal{F}_{t\to\xi} \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in M_{1/w}^{(p_3',q_3'),(q_4',p_4')}$ for $f \in M_{w_1}^{p_1,q_1}$ and $g \in M_{1/w_2}^{p_2',q_2'}$. Note that replacing $\mathfrak{g}$ by any other test function in (2) leads to a norm equivalent to $\|\cdot\|_{M_w^{p,q}}$, and we choose to show that for $\Psi = \mathcal{F}_{t\to\xi}\mathfrak{G}$, $\mathfrak{G}(x,\xi) = \mathfrak{g}(x)\mathfrak{g}(x-t)$, we have that

$$\big\|\tfrac{1}{w} V_\Psi \mathcal{F}_{t\to\xi} \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big\|_{L^{p_3',q_3',q_4',p_4'}}. \tag{18}$$

is bounded by the left hand side in (10) for $f \in M_{w_1}^{p_1,q_1}$ and $g \in M_{1/w_2}^{p_2',q_2'}$. Note that as

$$\big|\tfrac{1}{w} V_\Psi \mathcal{F}_{t\to\xi} \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big|(x,\xi,\nu,t) = \big|\tfrac{1}{w} V_{\mathfrak{G}} \overline{f}\otimes g \circ \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big|(x,t,\nu,\xi), \quad x,\xi,t,\nu \in \mathbb{R}^d,$$

the boundedness follows from an adjustment of the order of exponentiation and integration in (10). Minkowski's integral inequality, namely,

$$\Big\|\int |w(x,\cdot)\, F(x,\cdot)|^p\, dx\Big\|_{L^{\frac{q}{p}}} \leq \int \Big(\int |w(x,y)\, F(x,y)|^q\, dy\Big)^{\frac{p}{q}} dx$$

implies that $\|w(x,y)f(x,y)\|_{L^{p,q}} \leq \|w(y,x)f(y,x)\|_{L^{qp}}$ if $p \leq q$. Hence, if $q_4' \leq p_4'$ and $q_3' \leq p_4'$, then we can move the $t$-integral in between the $x$-integral and the $\xi$-integral and obtain that (18) is bounded by the left hand side of (10).

We now prepare to apply Lemma 2.8. Observe that if we assume

$$p_4', p_1, p_2' \geq p_3', \quad q_4', q_1, q_2' \geq q_3', \quad p_4' \geq q_3', q_4', \tag{19}$$

and

$$\frac{1}{p_1} + \frac{1}{p_2'} = \frac{1}{p_3'} + \frac{1}{p_4'} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2'} = \frac{1}{q_3'} + \frac{1}{q_4'},$$

that is, $p_4, p_1', p_2 \leq p_3$ and $q_4, q_1', q_2 \leq q_3$ and $p_4 \leq q_3, q_4$, and

$$\frac{1}{p_1} + 1 - \frac{1}{p_2} = 1 - \frac{1}{p_3} + 1 - \frac{1}{p_4} \quad \text{and} \quad \frac{1}{q_1} + 1 - \frac{1}{q_2} = 1 - \frac{1}{q_3} + 1 - \frac{1}{q_4},$$

the latter being

$$\frac{1}{p_1} - \frac{1}{p_2} = 1 - \frac{1}{p_3} - \frac{1}{p_4} \quad \text{and} \quad \frac{1}{q_1} - \frac{1}{q_2} = 1 - \frac{1}{q_3} + \frac{1}{q_4},$$

and

$$1+\frac{1}{p_2}=\frac{1}{p_1}+\frac{1}{p_3}+\frac{1}{p_4} \quad \text{and} \quad 1+\frac{1}{q_2}=\frac{1}{q_1}+\frac{1}{q_3}+\frac{1}{q_4}.$$

Hence, we obtain (7) if (19) is satisfied. Note that for $\widetilde{p}\leq p$ and $\widetilde{q}\leq q$ we have $M_w^{\widetilde{p},\widetilde{q}}$ embeds continuously in $M_w^{p,q}$ (see, for example, Theorem 12.2.2 in [Grö01]). Hence, (7) remains true if we decrease $p_1,p_3,p_4$ and $q_1,q_3,q_4$, and/or increase $p_2$ and $q_2$. We conclude that (7) holds if (6) and $p_4\leq q_3,q_4$ are satisfied. □

**Remark 2.9** Note that for Hilbert–Schmidt operators, we have

$$\|H\|_{HS}=\|\kappa_H\|_{L^2}=\|h_H\|_{L^2}=\|\sigma_H\|_{L^2}=\|\eta_H\|_{L^2}, \tag{20}$$

a fact which is helpful to obtain norm inequalities of the form (1). But when considering modulation space norms for operator symbols, the chain of equalities (20) fails to hold. For example, we have

$$|\langle h_H,\pi(x,t,\nu,\xi)\mathfrak{g}\rangle|=|\langle\sigma_H,\pi(x,\xi,\nu,t)\mathfrak{g}\rangle|=|\langle\eta_H,\pi(t,\nu,\xi,x)\mathfrak{g}\rangle|,$$

but due to the implicitly given order of exponentiation and integration,

$$\|h_H\|_{M^{(p_1,p_2),(q_1,q_2)}}\not\asymp\|\sigma_H\|_{M^{(p_1,q_2),(q_1,p_2)}}\not\asymp\|\eta_H\|_{M^{(p_2,q_1),(q_2,p_1)}}, \quad H:\mathcal{S}(\mathbb{R}^d)\longrightarrow\mathcal{S}'(\mathbb{R}^d).$$

Consequently, when defining a modulation space type norm on sets of pseudodifferential operators, one can base it on either $h_H$, $\sigma_H$, or $\eta_H$, each choice leading to different operator spaces and norms. Lemma 2.8 gives a hint that it may be advantageous to define operator modulation spaces $OM^{p_1,p_2,q_1,q_2}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ through finiteness of the norm

$$\|H\|_{OM_w^{p_1,p_2,q_1,q_2}}=\Big(\int\Big(\int\Big(\int\Big(\int\big|V_{\mathfrak{g}}^s\sigma_H(x,t,\xi,\nu)\,w(x,t,\xi,\nu)\big|^{p_1}dx\Big)^{\frac{p_2}{p_1}}dt\Big)^{\frac{q_1}{p_2}}d\xi\Big)^{\frac{q_2}{q_1}}d\nu\Big)^{\frac{1}{q_2}},$$

where the *symplectic short-time Fourier transform* $V^s$ with respect to the window function $\mathfrak{g}\in\mathcal{S}(\mathbb{R}^{2d})$ is given by

$$V_{\mathfrak{g}}^sF(x,t,\xi,\nu)=\mathcal{F}^s\big(F\cdot\overline{T_{x,\xi}\mathfrak{g}}\big)(t,\nu), \quad F\in\mathcal{S}'(\mathbb{R}^{2d}).$$

This choice of order of exponentiation and integration respects arranging the *time* variables ahead of the *frequency* variables, while listing first the *absolute time* variable $x$ and then the *time-shift* variable $t$, respectively, we first list the *absolute frequency* variable $\xi$ and then the *frequency-shift* variable $\nu$. More importantly, with this choice, we have

$$\|Hf\|_{M^{p_2,q_2}}\leq C\|H\|_{OM^{p_3,p_4,q_3,q_4}}\,\|f\|_{M^{p_1,q_1}}, \quad H\in OM^{p_1,p_2,q_1,q_2}(M^{p_1,q_1}(\mathbb{R}^d),M^{p_2,q_2}(\mathbb{R}^d))$$

for all $1\leq p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4\leq\infty$ satisfying (6).

For simplicity of terminology, we avoid the use of operator modulation spaces and symplectic short-time Fourier transforms in the following. Lemma 2.3 implies that this does not lead to a loss of generality in case of the here considered operator Paley–Wiener spaces.

## 3 Sampling and reconstruction in operator Paley–Wiener spaces

We introduce *operator Paley–Wiener spaces.*

**Definition 3.1** *For $1 \leq p,q \leq \infty$ and a moderate weight $w$ on $\mathbb{R}^{2d}$, operator Paley–Wiener spaces are given by*

$$OPW_w^{p,q}(M) = \{H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d) : \ \operatorname{supp} \mathcal{F}^s \sigma_H \subseteq M \ \text{and} \ \sigma_H \in L_w^{p,q}(\mathbb{R}^{2d})\}.$$

*$OPW_w^{p,q}(M)$ is a Banach space with norm $\|H\|_{OPW_w^{p,q}} = \|\sigma_H\|_{L_w^{p,q}}$. If $w \equiv 1$ and $p = q = 2$ then we simply write $OPW(M) = \big\{H \in HS(L^2(\mathbb{R}^d)) : \quad \operatorname{supp} \mathcal{F}_s \sigma_H \subseteq M\big\}$.*

Note that, as illustrated in Corollary 3.7 and Example 3.8 below, it is appropriate to choose $OPW_w^{p,\infty}(M)$, respectively $OPW_w^{\infty,q}(M)$, when considering multiplication respectively convolution operators. Moreover, observe that $OPW_w^{\infty,\infty}(M)$ consists of all operators in the weighted Sjöstrand class with Kohn–Nirenberg symbol bandlimited to $M$ [Grö06, Sjö94, Sjö95, Str06].

**Remark 3.2** In [Hör07], Hörmander considers pseudodifferential operators with Kohn–Nirenberg symbol in

$$S_{\rho,\delta}^m = \big\{\sigma \in C^\infty(\mathbb{R}^2 d) : \ \big|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)\big| \leq C_{\alpha,\beta}(1 + \|\xi\|_2)^{m-\rho(\alpha_1+\ldots+\alpha_d)+\delta(\beta_1+\ldots+\beta_d)}, \quad \alpha,\beta \in \mathbb{N}_0^d\big\}$$

where $m \in \mathbb{R}$, $0 < \rho \leq 1$, and $0 \leq \delta < 1$. Clearly, if $\operatorname{supp} \mathcal{F}^s \sigma \subseteq M$ and $\sigma \in S_{\rho,\delta}^m$, then $L_\sigma \in OPW_{1\otimes w_s}^{\infty,\infty}(M)$ if $s \leq -m$ and $L_\sigma \in OPW_{1\otimes w_s}^{\infty,q}(M)$ if $(m+s)q < -1$.

**Theorem 3.3** *Let $1 \leq p,q \leq \infty$ and $w$ moderate on $\mathbb{R}^2$. For $M$ compact exists $C > 0$ with*

$$\|Hf\|_{M_w^{p,q}} \leq C\, \|\sigma_H\|_{L_w^{p,q}}\, \|f\|_{M^{\infty,\infty}}, \quad H \in OPW_w^{p,q}(M),\ f \in M^{\infty,\infty}(\mathbb{R}^d).$$

*Consequently, any $H \in OPW_w^{p,q}(M)$ extends to a bounded operator mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,q}(\mathbb{R}^d)$.*

*Proof.* Set $\omega(x,\xi,\nu,t) = w(x,\xi+\nu)$ and choose $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}\widehat{\varphi} \subseteq [-1,1]^d$. Then we use Lemma 2.3 and $\operatorname{supp} V_{\varphi\otimes\varphi}\sigma_H \subseteq \mathbb{R}^{2d} \times M + [-1,1]^{2d}$, hence, $\omega \asymp w\otimes 1$ on $\operatorname{supp} V_{\varphi\otimes\varphi}\sigma_H$, to obtain

$$\|\sigma_H\|_{L_w^{p,q}} \asymp \|\sigma_H\|_{M_{w\otimes 1}^{(p,q),(1,1)}} \asymp \|w\otimes 1\, V_{\varphi\otimes\varphi}\sigma_H\|_{L^{(p,q),(1,1)}} \asymp \|\omega\, V_{\varphi\otimes\varphi}\sigma_H\|_{L^{(p,q),(1,1)}} \asymp \|\sigma_H\|_{M_\omega^{(p,q),(1,1)}}.$$

An application of Theorem 2.7 with $p_1 = q_1 = \infty$, that is $p_1' = q_1' = 1$, $p_2 = p_3 = p$, $q_2 = q_3 = q$, and $p_4, q_4 = 1$ concludes the proof. □

In the following, we set $Q_T = [0,T_1)\times\ldots\times[0,T_d)$ for $T = (T_1,\ldots,T_d) \in (\mathbb{R}^+)^d$ and $R_\Omega = [-\frac{\Omega_1}{2},\frac{\Omega_1}{2})\times\ldots\times[-\frac{\Omega_d}{2},\frac{\Omega_d}{2})$ for $\Omega = (\Omega_1,\ldots,\Omega_d) \in (\mathbb{R}^+)^d$.

**Theorem 3.4** *Let $1 \leq p,q \leq \infty$ and let $w = w_1\otimes w_2$ be moderate on $\mathbb{R}^{2d}$. Let $T,\Omega \in (\mathbb{R}^+)^d$ satisfy $T_m\Omega_m < 1$, $m = 1,\ldots,d$. Let $\Lambda = T_1\mathbb{Z}\times\ldots\times T_d\mathbb{Z}$ and choose $s \in M^{1,1}(\mathbb{R}^d)$ with $\operatorname{supp}\widehat{s} \subseteq R_{1/T}$ and $\widehat{s} \equiv 1$ on $R_\Omega$. Then*

$$\|H\sum_{\lambda\in\Lambda}\delta_\lambda\|_{M_w^{p,q}} \asymp \|H\|_{OPW_w^{p,q}}, \quad H \in OPW_w^{p,q}(Q_T\times R_\Omega), \tag{21}$$

*and any $H \in OPW_w^{p,q}(Q_T\times R_\Omega)$ can be reconstructed by means of*

$$\kappa_H(x+t,x) = \chi_{Q_T}(t)\sum_{\lambda\in\Lambda}\big(H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\big)(t+\lambda)\, s(x-\lambda). \tag{22}$$

*with convergence in $OPW_w^{p,q}(\mathbb{R}^{2d})$ for $1 \leq p,q < \infty$ and weak-$*$ convergence else.*

*Proof.* We shall show (22). The norm equivalence (21) can be shown by adopting the steps of the proof of Theorem 4.6.

For $\Lambda = T_1\mathbb{Z}\times\ldots\times T_d\mathbb{Z}$, we consider the Zak transform given by

$$Z_\Lambda f(t,\nu)=\sum_{\lambda\in\Lambda} f(t-\lambda)\,e^{2\pi i\lambda\nu},\quad (t,\nu)\in Q_T\times R_{\frac{1}{T}}.$$

Note $\big(H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\big)(x)=\langle\kappa_H(x,\cdot),\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\rangle=\sum_{\lambda'\in\Lambda}\kappa_H(x,\lambda')=\sum_{\lambda'\in\Lambda}h_H(x,x-\lambda')$. We consider first $h_H\in M^{1,1}(\mathbb{R}^{2d})$ and use the Tonelli–Fubini Theorem and the Poisson Summation Formula [Grö01], page 250, to obtain for $(t,\nu)\in Q_{T,\frac{1}{T}}$

$$\begin{aligned}
Z_\Lambda\circ H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}(t,\nu) &= \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda} h_H(t-\lambda,t-\lambda-\lambda')e^{2\pi i\lambda\nu}\\
&= \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}\int \eta_H(t-\lambda-\lambda',\nu')e^{2\pi i(t-\lambda)\nu'}\,d\nu'\,e^{2\pi i\lambda\nu}\\
&= \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}\int \eta_H(t-\lambda-\lambda',\nu''+\nu)e^{2\pi i((t-\lambda)(\nu+\nu'')+\lambda\nu)}\,d\nu''\\
&= e^{2\pi it\nu}\sum_{\lambda''\in\Lambda}\sum_{\lambda\in\Lambda}\int \eta_H(t-\lambda'',\nu+\nu'')e^{2\pi it\nu''}\,e^{-2\pi i\lambda\nu''}\,d\nu''\\
&= e^{2\pi it\nu}\sum_{\lambda''\in\Lambda}\sum_{\lambda\in\Lambda^\perp}\eta_H(t-\lambda'',\nu+\lambda)e^{2\pi it\lambda}\\
&= \sum_{\lambda''\in\Lambda}\sum_{\lambda\in\Lambda^\perp}\eta_H(t-\lambda'',\nu-\lambda)e^{2\pi it(\nu-\lambda)},
\end{aligned}$$

where $\Lambda^\perp=\{\lambda\in\mathbb{R}^d : e^{2\pi i\lambda\lambda'}=1 \text{ for all } \lambda'\in\Lambda\}=\frac{1}{T_1}\mathbb{Z}\times\ldots\times\frac{1}{T_d}\mathbb{Z}$ is the dual lattice of $\Lambda$.

This leads directly to (22) since

$$\begin{aligned}
\int\ &\chi_{Q_T}(t)\ \sum_{\lambda\in\Lambda}\big(H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\big)(t+\lambda)\,s(x-\lambda)\ e^{-2\pi i\nu x}\,dx\\
&= \chi_{Q_T}(t)\sum_{\lambda\in\Lambda}\big(H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\big)(t+\lambda)\int s(x-\lambda)\ e^{-2\pi i\nu x}\,dx\\
&= \chi_{Q_T}(t)\sum_{\lambda\in\Lambda}\big(H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\big)(t+\lambda)\ e^{-2\pi i\lambda\nu}\,\widehat{s}(\nu)\\
&= \chi_{Q_T}(t)\widehat{s}(\nu)\ \left(Z_\Lambda\big(H\sum_{\lambda'\in\Lambda}\delta_{\lambda'}\big)\right)(t,\nu)\\
&= \eta_H(t,\nu)e^{2\pi i\nu t}=\int h_H(x,t)\ e^{-2\pi i\nu(x-t)}\,dx=\int h_H(x+t,t)\ e^{-2\pi i\nu x}\,dx.
\end{aligned}$$

We can apply Lemma 2.3 to show that $\|H\|_{OPW^{p,q}_w}\asymp\|h_H\|_{M^{(p,1),(1,q)}_{\widetilde{w}}}$, $\widetilde{w}(x,t,\nu,\xi)=w(x,\xi)$, and validity of (22) for $h_H\in M^{(p,1),(1,q)}_{\widetilde{w}}(\mathbb{R}^{2d})$ follows then from the density of $M^{1,1}_{\widetilde{w}}(\mathbb{R}^{2d})$ in $M^{(p,1),(1,q)}_{\widetilde{w}}(\mathbb{R}^{2d})$. In case of $p=\infty$ or $q=\infty$ it follows from weak-$*$ density. $\square$

**Remark 3.5** If $p,q=2$, then we can alternatively choose $s=\chi_{R_\Omega}\in M^{2,2}(\mathbb{R}^d)=L^2(\mathbb{R}^d)$. This allows us to replace the inequality $T_m\Omega_m<1$ by $T_m\Omega_m\le 1$ in the hypothesis of Theorem 3.4.

Note that Theorem 3.4 and its proof generalize trivially to the following setting.

**Theorem 3.6** *Let $1 \leq p, q \leq \infty$ and $w = w_1 \otimes w_2$ be moderate on $\mathbb{R}^{2d}$. Let $A, B \subseteq \mathbb{R}^d$ be bounded, and let $\Lambda$ be a lattice such that $A$ is contained in a fundamental domain of $\Lambda$ and for some $\epsilon > 0$, $B + [-\epsilon, \epsilon)^d$ is contained in a bounded fundamental domain of $\Lambda^\perp = \{\lambda \in \mathbb{R}^d : e^{2\pi i \lambda \lambda'} = 1$ for all $\lambda' \in \Lambda\}$. Choose $s \in M^{1,1}(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{s} \subseteq B + [-\epsilon, \epsilon)^d$ and $\widehat{s} \equiv 1$ on $B$. Then*

$$\|H \sum_{\lambda \in \Lambda} \delta_\lambda\|_{M^{p,q}_w} \asymp \|H\|_{OPW^{p,q}_w}, \quad H \in OPW^{p,q}_w(A \times B),$$

*and any $H \in OPW^{p,q}_w(A \times B)$ can be reconstructed by means of*

$$\kappa_H(x+t, t) = \chi_A(t) \sum_{\lambda \in \Lambda} \big(H \sum_{\lambda' \in \Lambda} \delta_{\lambda'}\big)(t + \lambda)\, s(x - \lambda).$$

*with convergence in $OPW^{p,q}_w(A \times B)$ for $1 \leq p, q < \infty$ and weak-$*$ convergence else.*

Considering $OPW^{p,\infty}([0,T) \otimes [-\frac{\Omega}{2}, \frac{\Omega}{2}))$, we obtain the classical sampling theorem as corollary to Theorem 3.4.

**Corollary 3.7** *For $m \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, with $\operatorname{supp} \widehat{m} \subseteq [-\frac{\Omega}{2}, \frac{\Omega}{2})$ and $T$ with $T\Omega < 1$ choose $s \in M^{1,1}(\mathbb{R})$ with $\operatorname{supp} \widehat{s} \subseteq [-\frac{\Omega}{2}, \frac{\Omega}{2})$ and $\widehat{s} \equiv 1$ on $[-\frac{1}{2T}, \frac{1}{2T})$. Then*

$$\|m\|_{L^p} \asymp \|\, \{m(kT)\}\, \|_{l^p} \tag{23}$$

*and*

$$m(x) = \sum_{k \in \mathbb{Z}} m(kT)\, s(x - kT).$$

*Proof.* For $m \in L^p(\mathbb{R})$ with $\operatorname{supp} \widehat{m} \subseteq [-\frac{\Omega}{2}, \frac{\Omega}{2})$, we define the multiplication operator $M$ formally by $M : f \longrightarrow m \cdot f$, $f \in \mathcal{S}(\mathbb{R})$. We have

$$Mf(x) \;=\; m(x)f(x) = \int m(x)\delta_0(t) f(x-t)dt = \iint \delta_0(t)\widehat{m}(\nu) e^{2\pi i x \nu} f(x-t)\, dt\, d\nu\,.$$

Hence, $\delta_0 \otimes \widehat{m} = \eta_M = \mathcal{F}^s \sigma_M$, and, picking any $T$ with $T\Omega < 1$, we conclude $M \in OPW^{p,\infty}\big([0,T) \times [-\frac{\Omega}{2}, \frac{\Omega}{2})\big)$.

Theorem 3.4 implies that $H$ and therefore $m$ is fully recoverable from $M \sum_{k \in \mathbb{Z}} \delta_{kT} = \sum_{k \in \mathbb{Z}} m(kT) \delta_{kT}$, in fact, the reconstruction formula (22) reduces then to the classical reconstruction formula for functions:

$$\begin{aligned}
m(x)\delta_0(t) &= m(x+t)\delta_0(t) = \kappa_M(x+t, x) \overset{\text{Thm 3.4}}{=} \chi_{[0,T)}(t) \sum_{n \in \mathbb{Z}} \big(M \sum_{k \in \mathbb{Z}} \delta_{kT}\big)(t + nT)\, s(x - nT) \\
&= \chi_{[0,T)}(t) \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} m(kT) \delta_{kT}(t + nT) s(x - nT) \\
&= \chi_{[0,T)}(t) \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} m(kT) \delta_0(t - (kT - nT)) s(x - nT) \\
&= \begin{cases} 0, & \text{if } t \notin [0,T) \text{ or } t \notin \mathbb{Z}T; \\ \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} m(kT) \delta_0((n-k)T) s(x - nT), & \\ \quad = \sum_{k \in \mathbb{Z}} m(kT) s(x - kT), & \text{if } t = 0. \end{cases} \\
&= \delta_0(t) \sum_{k \in \mathbb{Z}} m(kT) s(x - kT)\,.
\end{aligned}$$

The norm equivalence in (23) is obtained by verifying that

$$\|m\|_{L^p} \asymp \|M\|_{OPW^{p,\infty}} \asymp \|M\sum_{n\in\mathbb{Z}}\delta_{nT}\|_{M^{p,\infty}} = \|\sum_{n\in\mathbb{Z}} m(nT)\delta_{nT}\|_{M^{p,\infty}} \asymp \|\{m(nT)\}_n\|_p,$$
$$m\in L^p(\mathbb{R}),\ \operatorname{supp}\widehat{m}\subseteq[-\frac{\Omega}{2},\frac{\Omega}{2}].$$

□

In addition to the application of Theorem 3.4 to multiplication operators, we consider now $OPW^{\infty,p}([0,T)\otimes[-\frac{\Omega}{2},\frac{\Omega}{2}))$ for convolution operators.

**Example 3.8** Time invariant operators are convolution operators, that is,

$$Hf(x) = h*f(x) = \int h(x-s)f(s)\,ds.$$

Such operators represent the classical example of operator identification/sampling namely, as $H\delta_0(x) = h(x)$, $H\delta_0$ determines $h$ and therefore $H$ completely. In the framework of operator sampling, we consider $h\in L^p(\mathbb{R})$ with $\operatorname{supp} h\subseteq[0,T]$. We have $\eta_H(t,\nu) = \mathcal{F}^s\sigma_H(t,\nu) = h(t)\delta_0(\nu)$ and $H\in OPW^{\infty,p}\big([0,T)\times[-\frac{1}{4T},\frac{1}{4T})\big)$. Moreover, with appropriate $s$ we may obtain $H\delta_0 = h$ from (22), as

$$\begin{aligned}
h(t) &= h_H(x,t) = h_H(x+t,t) \overset{\text{Thm 3.4}}{=} \chi_{[0,T)}(t)\sum_{n\in\mathbb{Z}}\big(H\sum_{k\in\mathbb{Z}}\delta_{kT}\big)(t+nT)\,s(x-nT)\\
&= \chi_{[0,T)}(t)\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} h(t-(kT-nT))\,s(x-nT) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\chi_{[0,T)}(t)h(t-(kT-nT))\,s(x-nT)\\
&= \sum_{n\in\mathbb{Z}} H\delta_0(t)s(x-nT) = H\delta_0(t)\sum_{n\in\mathbb{Z}} s(x-nT) = H\delta_0(t)\sum_{\ell\in\mathbb{Z}}\widehat{s}(\tfrac{\ell}{T})e^{2\pi ix\ell} = H\delta_0(t)\widehat{s}(0) = H\delta_0(t)\,.
\end{aligned}$$

The distributional spreading support of a time invariant operator is also indicated in Figure 1.

# 4 Necessary and sufficient conditions for operator sampling and identification

The aim of this section is to show that the applicability of sampling methods for operators depends solemnly on the size of the spreading support set $M$, that is, on the Jordan content of $M$ (see Definition 4.4 below). Our main result in this section, namely Theorem 4.6, though, only covers the case $d=1$, that is, operators $H:\mathcal{S}(\mathbb{R})\longrightarrow\mathcal{S}'(\mathbb{R})$. Possible means for generalizing Theorem 4.6 to operators $H:\mathcal{S}(\mathbb{R}^d)\longrightarrow\mathcal{S}'(\mathbb{R}^d)$ are briefly discussed in Section 5.

Before recalling the definition of Jordan domains and some of their properties, and before stating and proving Theorems 4.6 and 4.7, we will use a geometric approach to obtain a sufficient condition for the identifiability of $OPW^{p,p}_w(M)$ if $M = A(Q_T\times R_\Omega)+(t_0,\nu_0)\subseteq\mathbb{R}^{2d}$, $T,\Omega\in(\mathbb{R}^+)^d$, $T_m\Omega_m<1$, $m=1,\ldots,d$, and $A$ is a so-called symplectic matrix. Theorem 4.2 below generalizes Theorem 5.4 in [KP06].

**Definition 4.1** *The symplectic group $Sp(d,\mathbb{R})$ consists of those matrices $A\in SL(2d,\mathbb{R}) = \{A\in R^{2d\times 2d}:\ \det A = 1\}$ with $A^\star\begin{pmatrix}0 & -I_d\\ I_d & 0\end{pmatrix}A = \begin{pmatrix}0 & -I_d\\ I_d & 0\end{pmatrix}$, where $I_d$ is the $d\times d$ identity matrix.*

Note that $A\in Sp(d,\mathbb{R})$ if and only if $[A(x,\xi)^T, A(x',\xi')T] = [(x,\xi),(x',\xi')]$ where $[\cdot,\cdot]$ is the symplectic form defined in Section 2.

**Theorem 4.2** *Let $A \in Sp(d,\mathbb{R})$, $(t_0,\nu_0) \in \mathbb{R}^{2d}$, $1 \le p \le \infty$, and let $w$ be a moderate weight on $\mathbb{R}^{2d}$ with $w(A(x,\xi)^T) \le w(x,\xi)$. Then*

1. *$OPW_w^{p,p}(M)$ mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,p}(\mathbb{R}^d)$ is identifiable if and only if $OPW_w^{p,p}\big(AM+(t_0,\nu_0)\big)$ mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,p}(\mathbb{R}^d)$ is identifiable, and, consequently,*
2. *for $T,\Omega \in (\mathbb{R}^+)^d$ with $T_m\Omega_m < 1$, $m=1,\dots,d$, we have $OPW_w^{p,p}\big(A(Q_T{\times}R_\Omega)+(t_0,\nu_0)\big)$ mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,p}(\mathbb{R}^d)$ is identifiable.*

The proof of Theorem 4.2 is based on the representation theory of the Weyl-Heisenberg group. Here, we only outline the proof, the interested reader can import details from Section 5 in [KP06] or [Fol89, Grö01].

*Proof.* We will obtain the identifiability $OPW_w^{p,p}(AM)$ with $A \in Sp(2d,\mathbb{R})$ from the identifiability of $OPW_w^{p,p}(M)$ by using the canonical correspondence between elements in $OPW_w^{p,p}(AM)$ and elements in $OPW_w^{p,p}(M)$ which is given by a coordinate transformation in the spreading domain $\mathbb{R}^{2d} \supseteq M, AM$. In fact, Theorem 5.3 in [KP06] recalls that for $A \in Sp(d,\mathbb{R})$, there exists a unitary operators $O_A$ on $L^2(\mathbb{R}^d)$ with $\pi(A(t,\nu)) = O_A\pi(t,\nu){O_A}^*$, $t,\nu \in \mathbb{R}^d$. Such operators $O_A$, $A \in Sp(d,\mathbb{R})$ are called metaplectic operators, and they are intertwining operators for representations of the reduced Weyl–Heisenberg group that are unitarily equivalent to the Schrödinger representation [Fol89, Grö01]. Metaplectic operators are finite compositions of the Fourier transform, multiplication operators with multiplier $e^{-\pi i x^T C x}$ with $C$ selfadjoint, and normalized dilations $f \mapsto |\det D|^{\frac{1}{2}} f(Dx)$, $D$ invertible. They extend, respectively restrict, to isomorphisms on $M_w^{p,p}(\mathbb{R}^d)$, $1 \le p \le \infty$, if $w(A(x,\xi)^T) \le w(x,\xi)$ (see Theorem 7.4 in [FG92]).

The following formal calculations of operator valued integrals can be justified weakly for all $H \in OPW_w^{p,p}(AM)$. A similar computation can be made for $H \in OPW_w^{p,p}\big(M+(t_0,\nu_0)\big)$, combining both leads Theorem 4.2. We compute

$$\begin{aligned}
H &= \iint \eta_H(t,\nu)\, M_\nu T_t \; dt\, d\nu = \iint \eta_H(t,\nu)\ \pi(t,\nu)\, dt\, d\nu \\
&= \iint \eta_H(A(t,\nu)^T)\ \pi(A(t,\nu)^T)\ dt\, d\nu = \iint \eta_H(A(t,\nu)^T)\ O_A\pi(t,\nu){O_A}^*\ dt\, d\nu \\
&= O_A \iint \eta_{H_A}(t,\nu)\ \pi(t,\nu)\ dt\, d\nu\ {O_A}^* = O_A\ H_A\ {O_A}^*,
\end{aligned}$$

where $\eta_{H_A} = \eta_H \circ A$ and $H_A \in OPW_w^{p,p}(M)$. The identifiability of $OPW_w^{p,p}(M)$ with identifier $f_M \in M^{\infty,\infty}(\mathbb{R}^d)$ leads therefore to the identifiability of $OPW_w^{p,p}(AM)$ with identifier $f_{AM} = O_A f_M \in M^{\infty,\infty}(\mathbb{R}^d)$. In fact, we have

$$\begin{aligned}
\|H f_{AM}\|_{M_w^{p,p}} &= \|H\, O_A f_M\|_{M_w^{p,p}} \asymp \|{O_A}^* H O_A f_M\|_{M_w^{p,p}} \\
&= \|H_A f_M\|_{M_w^{p,p}} \asymp \|\sigma_{H_A}\|_{L_w^{p,p}} \asymp \|\sigma_H\|_{L_w^{p,p}} = \|H\|_{OPW_w^{p,p}}, \quad H \in OPW_w^{p,p}(AM).
\end{aligned}$$

□

**Remark 4.3** Theorem 4.2 is not an operator sampling result per se as not necessarily all $O_A$ map discretely supported distributions to discretely supported distributions. But Theorem 4.2 can be used to show that $OPW_w^{p,p}(M)$ permits operator sampling by showing that

1. $M \subseteq A\widetilde{M} + (t_0,\nu_0)$,
2. $A \in Sp(d,\mathbb{R})$,
3. $w(A(x,\xi)) \le w(x,\xi)$,

4. $OPW_w^{p,p}(\widetilde{M})$ permits operator sampling with sampling set $\{x_j\}$ and weights $\{c_j\}$, and

5. $O_A^* \sum c_j \delta_{x_j}$ is discretely supported.

Note also that the restriction to $p = q$ in Theorem 4.2 is necessary, as, for example, the Fourier transform is not an isomorphism on $M^{p,q}$ whenever $p \neq q$.

Theorem 4.2 relies on arguments based on symplectic geometry on phase space. As discussed above, Theorems 4.6 and 4.7 give a characterization for the identifiability of operators $H : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$ which does not rely on any geometric properties.

**Definition 4.4** *For $K, L \in \mathbb{N}$ set $R_{K,L} = [0, \frac{1}{K}) \times [0, \frac{K}{L})$ and*

$$\mathcal{U}_{K,L} = \Big\{ \bigcup_{j=1}^{J} \Big( R_{K,L} + (\tfrac{k_j}{K}, \tfrac{p_j K}{L}) \Big) : \; k_j, p_j \in \mathbb{Z}, J \in \mathbb{N} \Big\}.$$

*The inner content, respectively outer content, of a bounded set $M \subseteq \mathbb{R}^2$ is*

$$\mathrm{vol}^-(M) = \sup\{\mu(U) : U \subseteq M \text{ and } U \in \mathcal{U}_{K,L} \text{ for some } K, L \in \mathbb{N}\}, \tag{24}$$

*respectively*

$$\mathrm{vol}^+(M) = \inf\{\mu(U) : U \supseteq M \text{ and } U \in \mathcal{U}_{K,L} \text{ for some } K, L \in \mathbb{N}\}. \tag{25}$$

*Clearly, we have $\mathrm{vol}^-(M) \leq \mathrm{vol}^+(M)$. If $\mathrm{vol}^-(M) = \mathrm{vol}^+(M)$, then we say that $M$ is a Jordan domain with Jordan content $\mathrm{vol}(M) = \mathrm{vol}^-(M) = \mathrm{vol}^+(M)$.*

We collect some well known and useful facts on Jordan domains to illustrate their generality [Fol99].

**Proposition 4.5** *Let $M \subseteq \mathbb{R}^2$.*

1. *If $M$ is a Jordan domain, then $M$ is Lebesgue measurable with $\mu(M) = \mathrm{vol}(M)$.*

2. *If $M$ is Lebesgue measurable and bounded and its boundary $\partial M$ is a Lebesgue zero set, that is, $\mu(\partial M) = 0$, then $M$ is a Jordan domain.*

3. *If $M$ is open, then $\mathrm{vol}^-(M) = \mu(M)$ and if $M$ is compact, then $\mathrm{vol}^+(M) = \mu(M)$.*

4. *If $\mathcal{P} \subseteq \mathbb{N}$ is unbounded, then replacing the quantifier* "for some $L \in \mathbb{N}$" *with* "for some $L \in \mathcal{P}$" *in (24) and in (25) leads to equivalent definitions of inner and outer Jordan content.*

The second main result of this paper has been stated in simple form as Theorem 1.3, part 1, in Section 1. It also generalizes Theorem 1.1 in [PW06a].

**Theorem 4.6** *For $1 \leq p, q \leq \infty$ and $w = w_1 \otimes w_2$ moderate, the class $OPW_w^{p,q}(M)$ mapping $M^{\infty,\infty}(\mathbb{R})$ to $M_w^{p,q}(\mathbb{R})$ permits operator sampling if $\mathrm{vol}^+(M) < 1$. In fact, if $\mathrm{vol}^+(M) < 1$, then there exists $L > 0$ and a periodic sequence $\{c_n\}$ such that*

$$\|H \sum_{n \in \mathbb{Z}} c_n \delta_{\frac{n}{L}}\|_{M_w^{p,q}} \asymp \|H\|_{OPW_w^{p,q}}, \quad H \in OPW_w^{p,q}(M). \tag{26}$$

Theorem 4.6 is complemented by Theorem 4.7 which generalizes Theorem 1.1 in [PW06a] and Theorem 5.2, part 2, in [PW06b].

**Theorem 4.7** *Let $1 \leq p, q \leq \infty$ and $w$ subexponential. The class $OPW_w^{p,q}(M)$ mapping $M^{\infty,\infty}(\mathbb{R})$ to $M_w^{p,q}(\mathbb{R})$ is not identifiable if $\mathrm{vol}^-(M) > 1$, that is, for all $f \in M^{\infty,\infty}$ we have*

$$\|Hf\|_{M_w^{p,q}} \not\asymp \|H\|_{OPW_w^{p,q}} \quad H \in OPW_w^{p,q}(M)\,.$$

Theorem 4.6 is proven below. Subsequently, we outline the proof of Theorem 4.7 which employs elements of the proof of Theorem 1.1 in [PW06a] and Theorem 3.13 in [Pfa08b].

## 4.1 Proof of Theorem 4.6

The following observations are special cases of Theorem 4.2. They will be used in the following to reduce notational complexity.

**Proposition 4.8** *Let $1 \leq p, q \leq \infty$ and let $w$ be a moderate weight on $\mathbb{R}^2$.*

1. *$OPW_w^{p,q}(M)$ is identifiable by $f$ if and only if $OPW_w^{p,q}\big(M\cdot \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}\big)$ is identifiable by $D_a f : x \mapsto f(ax)$.*
2. *$OPW_w^{p,q}(M)$ is identifiable by $f$ if and only if $OPW_w^{p,q}\big(M + \lambda\big)$ is identifiable by $\pi(\lambda) f$.*

*Proof.* We shall proof *1.*, the proof of *2.* follows similarly. For $H \in OPW_w^{p,q}\big(M\cdot \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}\big)$, define $H_a \in OPW_w^{p,q}(M)$ by $\eta_{H_a} = \eta_H \circ \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$. Then $\sigma_{H_a} = \sigma_H \circ \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$ as well. We compute formally

$$\begin{aligned} \big(HD_a f\big)(\tfrac{x}{a}) &= \int \sigma_H(\tfrac{x}{a}, \xi)\, e^{2\pi i \frac{x}{a}\xi}\, \widehat{D_a f}(\xi)\, d\xi = \tfrac{1}{a} \int \sigma_H(\tfrac{x}{a}, \xi)\, e^{2\pi i x \frac{\xi}{a}}\, \widehat{f}(\tfrac{\xi}{a})\, d\xi \\ &= \int \sigma_H(\tfrac{x}{a}, a\xi)\, e^{2\pi i x \xi}\, \widehat{f}(\xi)\, d\xi = H_a f(x). \end{aligned}$$

Using standard density arguments, we conclude that

$$\|HD_a f\|_{M_w^{p,q}} \asymp \|D_{\frac{1}{a}} HD_a f\|_{M_w^{p,q}} = \|H_a f\|_{M_w^{p,q}} \asymp \|H_a\|_{OPW_w^{p,q}} \asymp \|H\|_{OPW_w^{p,q}}, \quad H \in OPW_w^{p,q}\big(M\cdot \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}\big)\,.$$

$\square$

Assume now that $\mathrm{vol}^+(M) < 1$. Applying Proposition 4.8, we assume, without loss of generality, that for $\delta > 0$ sufficiently small, $M + [-\frac{\delta}{2}, \frac{\delta}{2})^2 \subseteq [0,1) \times \mathbb{R}^+$. We choose $K$, $L \in \mathbb{N}$ with $L$ prime so that the following conditions are satisfied for some $0 < \epsilon < \delta$ and $M_\epsilon = M + [-\frac{\epsilon}{2}, \frac{\epsilon}{2})^2$

1. $\mathrm{vol}^+(M_\epsilon) < 1$,
2. $M_\epsilon \subseteq [0,1) \times [0, K)$,
3. $L \geq K$,
4. $M_\epsilon \subseteq U_M = \displaystyle\bigcup_{j=0}^{L-1} \Big(R_{K,L} + (\tfrac{m_j}{K}, \tfrac{n_j K}{L})\Big)$, $\;m_j, n_j \in \mathbb{Z}$, where $R_{K,L} = [0, \frac{1}{K}) \times [0, \frac{K}{L})$ and $(m_j, n_j) \neq (m_{j'}, n_{j'})$ if $j \neq j'$.

Note that $1 = \mathrm{vol}(U_M)$.

The following result from [LPW05] is a key component of our proof of Theorem 4.6. In fact, if the restriction to $L$ prime below could be weakened, then we would obtain a generalization of Theorem 4.6 to higher dimensions (see Section 5).

**Theorem 4.9** *For $c \in \mathbb{C}^L$ define $\pi(k,\ell)c$ by $(\pi(k,\ell)c)_j = c_{j-k}\, e^{2\pi i \frac{j\ell}{L}}$, $k,\ell = 0,\dots,L-1$, where $j-k$ is understood modulus $L$. If $L$ is prime, then for almost every $c \in \mathbb{C}^L$, the vectors in $\mathcal{G}_c = \{\pi(k,\ell)c\}_{k,\ell=0,\dots,L-1}$ are in general linear position, that is, any matrix composed of $L$ vectors of $\mathcal{G}_c$ is invertible.*

**Remark 4.10** Theorem 4.9 can be reformulated as a matrix identification result with identifier $c$ [KPR08]. The use of algorithms based on basis pursuit to determine a matrix $M$ from $Mc$ efficiently is discussed in [PRT08, PR09].

We now choose as $c \in l^\infty(\mathbb{Z})$ the periodic extension of a vector $(c_0,\dots,c_{L-1})$ which satisfies the conclusions of Theorem 4.9. In the following, we shall show that $\kappa_H$ can be recovered from $Hg$ with $g = \sum_{k\in\mathbb{Z}} c_k \delta_{\frac{k}{L}} \in M^{\infty,\infty}(\mathbb{R})$.

For simplicity, we shall assume first that $\kappa_H \in M^{1,1}(\mathbb{R}^2)$. This additional assumption implies that for $g \in M^{\infty,\infty}(\mathbb{R})$, we have $Hg \in M^{1,1}(\mathbb{R})$ [PW06b]. This enables us to switch the order of integration in many of the following computations. After the necessary computations are completed, we shall extend our result using standard density arguments.

Choose nonnegative $\varphi \in C_c^\infty(\mathbb{R})$ with $\int \varphi(x)\,dx = 1$ and $\operatorname{supp}\varphi \subseteq [-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. We shall consider the case $1 \le p < \infty$ only, the case $p = \infty$ requires only the usual adjustments.

Note that

$$\begin{aligned}
\left|V_\varphi Hg(t+\tfrac{n}{K},\nu)\right| &= \Big|\int \sum_{k\in\mathbb{Z}} c_k\, h(x, x+\tfrac{k}{K})\, e^{-2\pi i \nu x} \varphi(x-(t+\tfrac{n}{K})\,dx\Big| \\
&= \Big|\int \sum_{k\in\mathbb{Z}} c_k\, h(x+\tfrac{n}{K}, x+\tfrac{n}{K}+\tfrac{k}{K})\, e^{-2\pi i \nu \frac{n}{K}}\, \overline{\pi(t,\nu)\varphi(x)}\,dx\Big| \\
&= \Big|\int \sum_{k\in\mathbb{Z}} c_k\, h(x+\tfrac{n}{K}, x+\tfrac{n}{K}+\tfrac{k}{K})\, \overline{\pi(t,\nu)\varphi(x)}\,dx\Big|.
\end{aligned}$$

Set $w = \sum_{m\in\mathbb{Z}} w_1(\frac{mL}{K})\, \chi_{[\frac{mL}{K}, \frac{(m+1)L}{K})}$ and observe that $w \asymp w_1$. Then

$$\begin{aligned}
\|Hg\|_{M^{p,q}_w} &= \|V_\varphi Hg\|_{L^{p,q}_w} = \Big\|\Big(\sum_{n\in\mathbb{Z}} \int_{\frac{n}{K}}^{\frac{n+1}{K}} |V_\varphi Hg(t,\nu) w_1(t)|^p\Big)^{\frac{1}{p}}\,dt\Big\|_{L^q_{w_2}} \\
&= \Big\|\Big(\sum_{n\in\mathbb{Z}} \int_0^{\frac{1}{K}} |V_\varphi Hg(t+\tfrac{n}{K},\nu) w_1(t+\tfrac{n}{K})|^p\Big)^{\frac{1}{p}}\,dt\Big\|_{L^q_{w_2}} \\
&= \Big\|\Big(\sum_{n\in\mathbb{Z}} \int_0^{\frac{1}{K}} |V_\varphi Hg(t+\tfrac{n}{K},\nu) w(t+\tfrac{n}{K})|^p\Big)^{\frac{1}{p}}\,dt\Big\|_{L^q_{w_2}}.
\end{aligned}$$

We set $\psi = \chi_{[-\frac{\epsilon}{2},\frac{K}{L}+\frac{\epsilon}{2})} * \varphi$ and observe that $\psi(\nu')T_\omega\varphi(\nu') = T_\omega\varphi(\nu')$ for $\omega \in [0,\frac{K}{L})$, a fact that will be used to drop $\psi$ in (31) below. We compute for $t \in [0,\frac{1}{K})$

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} |w(t+\tfrac{n}{K})|^p \Big| V_\varphi Hg(t+\tfrac{n}{K},\nu)\Big|^p \\
=\ & \sum_{n\in\mathbb{Z}} |w(t+\tfrac{n}{K})|^p \Big| \int \sum_{k\in\mathbb{Z}} c_k\, h_H(x+\tfrac{n}{K}, x+\tfrac{n}{K}+\tfrac{k}{K})\overline{\pi(t,\nu)\varphi(x)}\, dx\Big|^p \\
=\ & \sum_{n\in\mathbb{Z}} |w(t+\tfrac{n}{K})|^p \Big| \int \sum_{k\in\mathbb{Z}} c_{n-k}\, h_H(x+\tfrac{n}{K}, x+\tfrac{k}{K})\overline{\pi(t,\nu)\varphi(x)}\, dx\Big|^p \\
=\ & \sum_{j=0}^{L-1}\sum_{m\in\mathbb{Z}} |w_1(\tfrac{mL}{K})|^p \Big| \int \sum_{k\in\mathbb{Z}} c_{j-k}\, h_H(x+\tfrac{mL+j}{K}, x+\tfrac{k}{K})\overline{\pi(t,\nu)\varphi(x)}\, dx\Big|^p \\
=\ & \sum_{j=0}^{L-1} \Big\| \sum_{m\in\mathbb{Z}} \int \sum_{k\in\mathbb{Z}} c_{j-k}\, h_H(x+\tfrac{mL+j}{K}, x+\tfrac{k}{K})\overline{\pi(t,\nu)\varphi(x)}\, dx \\
&\qquad\qquad V_\varphi(M_{-\frac{mL+j}{K}}\psi)(x',\xi')\Big\|^p_{L^p_{1\otimes w}} \qquad (27)\\
\asymp\ & \sum_{j=0}^{L-1} \Big\| \int\int \sum_{k\in\mathbb{Z}} c_{j-k}\Big(\sum_{m\in\mathbb{Z}} h_H(x+\tfrac{mL+j}{K}, x+\tfrac{k}{K})e^{-2\pi i\nu'\frac{mL+j}{K}}\Big) \\
&\qquad\qquad \overline{\pi(t,\nu)\varphi(x)}\ \psi(\nu')\,\widehat{\overline{\pi}}(x',\xi')\varphi(\nu')\, d\nu'\, dx\Big\|^p_{L^p_{1\otimes w_1}}, \qquad (28)
\end{aligned}
$$

where we used that $M_{-\frac{mL+j}{K}}\psi$ is an $l^p_{\widetilde{w}}$–Riesz basis in the Banach space $M^{p,p}_{1\otimes w_1}(\mathbb{R})$ to obtain (27), that is, we used

$$
\|\sum_{m\in\mathbb{Z}} a_m\|_{l^p_{w_1}} \asymp \|\sum_{m\in\mathbb{Z}} a_m M_{-\frac{mL+j}{K}}\psi\|_{M^{p,p}_{w_1}} = \|\sum_{m\in\mathbb{Z}} a_m V_\varphi M_{-\frac{mL+j}{K}}\psi\|_{L^p_{w_1}}.
$$

Note that

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}} h_H(x+\tfrac{mL+j}{K}, x+\tfrac{k}{K})e^{-2\pi i\nu'\frac{mL+j}{K}} = \sum_{m\in\mathbb{Z}}\int \eta_H(x+\tfrac{k}{K},\xi')e^{2\pi i(x+\frac{mL+j}{K})\xi'}\, d\xi' e^{-2\pi i\frac{mL+j}{K}\nu'} \\
=\ & \sum_{m\in\mathbb{Z}}\int \eta_H(x+\tfrac{k}{K},\xi')e^{2\pi i x\xi'}e^{2\pi i\frac{mL+j}{K}(\xi'-\nu')}\, d\xi' \\
=\ & \sum_{m\in\mathbb{Z}}\int \eta_H(x+\tfrac{k}{K},\nu'+\xi')e^{2\pi i x(\nu'+\xi')}e^{2\pi i\frac{j}{K}\xi'}e^{2\pi i\frac{mL}{K}\xi'}\, d\xi' \\
=\ & \sum_{\ell\in\mathbb{Z}} \eta_H(x+\tfrac{k}{K},\nu'+\tfrac{\ell K}{L})e^{2\pi i x(\nu'+\frac{\ell K}{L})}e^{2\pi i\frac{j\ell}{L}} \qquad (29)\\
=\ & \sum_{\ell\in\mathbb{Z}} \eta_H(x+\tfrac{k}{K},\nu'+\tfrac{\ell K}{L})e^{2\pi i(x+\frac{k}{K})(\nu'+\frac{\ell K}{L})}e^{-2\pi i(\frac{k}{K}\nu'+\frac{k\ell}{L})}e^{2\pi i\frac{j\ell}{L}} \\
=\ & \sum_{\ell\in\mathbb{Z}} \widetilde{\eta_H}(x+\tfrac{k}{K},\nu'+\tfrac{\ell K}{L})e^{2\pi i\frac{(j-k)\ell}{L}}e^{-2\pi i\frac{k}{K}\nu'}. \qquad (30)
\end{aligned}
$$

We have applied the Poisson Summation Formula to obtain (29). Moreover, we chose $\widetilde{\eta_H}(x',\nu') = \eta_H(x',\nu')e^{2\pi i x'\nu'}$ in (30).

After substituting (30) into (28), we integrate with respect to $t$ on $[0,\frac{1}{K})$ to obtain

$$
\begin{aligned}
&\int_0^{\frac{1}{K}} \sum_{n\in\mathbb{Z}} \big|w_1(t+\tfrac{n}{K},\nu)\big|^p \, \big|V_\varphi Hg(t+\tfrac{n}{K},\nu)\big|^p \, dt \\
\asymp\ & \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \| \int\int \sum_{k\in\mathbb{Z}} c_{j-k} \big( \sum_{\ell\in\mathbb{Z}} \widetilde{\eta}_H(x+\tfrac{k}{K},\nu'+\tfrac{\ell K}{L}) e^{2\pi i \frac{(j-k)\ell}{L}} e^{-2\pi i \frac{k}{K}\nu'} \big) \\
& \qquad\qquad\qquad \overline{\pi(t,\nu)\varphi(x)}\ \psi(\nu')\, \overline{\pi(\xi',x')\varphi(\nu')}\, d\nu'\, dx \big\|^p_{L^p_{1\otimes w_1}} \, dt \\
=\ & \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint | \int\int \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k}\, e^{2\pi i \frac{(j-k)\ell}{L}} \widetilde{\eta}_H(x+\tfrac{k}{K},\nu'+\tfrac{\ell K}{L}) \\
& \qquad\qquad\qquad e^{-2\pi i \frac{k}{K}\nu'}\, \overline{\pi(t,\nu)\varphi(x)}\ \psi(\nu')\, \overline{\pi(\xi',x')\varphi(\nu')}\, d\nu'\, dx\, w_1(x')\big|^p \, dx'\, d\xi'\, dt \\
\geq\ & \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} | \int \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k}\, e^{2\pi i \frac{(j-k)\ell}{L}} \\
& \qquad \int e^{-2\pi i \nu' \frac{k}{K}}\, \widetilde{\eta_H}(x+\tfrac{k}{K},\nu'+\ell\tfrac{K}{L})\, \overline{\pi(t,\nu)\varphi(x)}\, \psi(\nu')\, \overline{\pi(\xi',x')\varphi(\nu')}\ d\nu'\, dx\, w_1(x')\big|^p \, d\xi'\, dx'\, dt \\
=\ & \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} | \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k}\, e^{2\pi i \frac{(j-k)\ell}{L}} \\
& \qquad \iint e^{-2\pi i \nu' \frac{k}{K}}\, \widetilde{\eta_H}(x+\tfrac{k}{K},\nu'+\ell\tfrac{K}{L})\, \overline{\pi(t,\nu)\varphi(x)}\, \overline{\pi(\xi',x')\varphi(\nu')}\ dx\, d\nu'\, w_1(x')\big|^p \ d\xi'\, dx'\, dt \qquad (31)\\
=\ & \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} | \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k}\, e^{2\pi i \frac{(j-k)\ell}{L}} \\
& \qquad V_{\varphi\otimes\varphi}\widetilde{\eta_H}(t+\tfrac{k}{K},\xi'+\tfrac{\ell K}{L},\nu,x'+\tfrac{k}{K})\, e^{2\pi i \frac{k}{K}\nu}\, e^{2\pi i \frac{\ell k}{L}}\, e^{2\pi i x' \frac{\ell K}{L}}\, w_1(x')\big|^p \ d\xi'\, dx'\, dt \\
=\ & \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \sum_{j=0}^{L-1} | \sum_{j'=0}^{L} c_{j-k_{j'}}\, e^{2\pi i \frac{j\ell_{j'}}{L}} \\
& \qquad V_{\varphi\otimes\varphi}\widetilde{\eta_H}(t+\tfrac{k_{j'}}{K},\xi'+\tfrac{\ell_{j'}K}{L},\nu,x'+\tfrac{k_{j'}}{K})\, e^{2\pi i \frac{k_{j'}}{K}\nu}\, e^{2\pi i x' \frac{\ell_{j'}K}{L}}\, w_1(x')\big|^p \ d\xi'\, dx'\, dt \qquad (32)\\
\asymp\ & \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \sum_{j'=0}^{L} \big|V_{\varphi\otimes\varphi}\widetilde{\eta_H}(t+\tfrac{k_{j'}}{K},\xi'+\tfrac{\ell_{j'}K}{L},\nu,x'+\tfrac{k_{j'}}{K})\, e^{2\pi i \frac{k_{j'}}{K}\nu}\, e^{2\pi i x' \frac{\ell_{j'}K}{L}}\, w_1(x'+\tfrac{k_{j'}}{K})\big|^p \, d\xi'\, dx'\, dt \\
=\ & \sum_{j'=0}^{L} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \big|V_{\varphi\otimes\varphi}\widetilde{\eta_H}(t+\tfrac{k_{j'}}{K},\xi'+\tfrac{\ell_{j'}K}{L},\nu,x')\, w_1(x')\big|^p \, d\xi'\, dx'\, dt \\
=\ & \sum_{j'=0}^{L} \int_{\frac{k_{j'}}{K}}^{\frac{(k_{j'}+1)}{K}} \iint_{\frac{l_{j'}K}{L}}^{\frac{(l_{j'}+1)}{L}} \big|V_{\varphi\otimes\varphi}\widetilde{\eta}(t,\xi',\nu,x')\, w_1(x')\big|^p \, d\xi'\, dx'\, dt\,.
\end{aligned}
$$

To obtain (32) we used that $V_{\varphi\otimes\varphi}\widetilde{\eta} \subseteq [0,1)\times[0,K)$. Moreover, we used

$$
\begin{aligned}
&\iint e^{-2\pi i\nu'\frac{k}{K}}\,\widetilde{\eta_H}(x+\tfrac{k}{K},\nu'+l\tfrac{K}{L})\,\overline{\pi(t,\nu)\varphi(x)}\,\overline{\pi(\xi',x')\varphi(\nu')}\;dx\,d\nu'\\
=\;&\iint e^{-2\pi i\nu'\frac{k}{K}}\,\widetilde{\eta_H}(x+\tfrac{k}{K},\nu'+l\tfrac{K}{L})\,e^{-2\pi ix\nu}\varphi(x-t)\,e^{-2\pi ix'\nu'}\varphi(\nu'-\xi')\;dx\,d\nu'\\
=\;&\iint \widetilde{\eta_H}(x,\nu')\,e^{-2\pi i\nu(x-\frac{k}{K})}\varphi(x-(t+\tfrac{k}{K}))\,e^{-2\pi i(\nu'-\frac{\ell K}{L})\frac{k}{K}}\,e^{-2\pi ix'(\nu'-\frac{\ell K}{L})}\,\varphi(\nu'-(\xi'+\tfrac{\ell K}{L}))\;dx\,d\nu'\\
=\;&V_{\varphi\otimes\varphi}\widetilde{\eta_H}(t+\tfrac{k}{K},\xi'+\tfrac{\ell K}{L},\nu,x'+\frac{k}{K})\,e^{2\pi i\frac{k}{K}\nu}\,e^{2\pi i\frac{\ell k}{L}}\,e^{2\pi ix'\frac{\ell K}{L}}\,.
\end{aligned}
$$

Using the fact that replacing now $\varphi\otimes\varphi$ by any other test functions leads to equivalent norms of the modulation space at hand, we obtain for real valued $\varrho\in S(\mathbb{R}^2)$, $\varrho(t,\nu)=1$ for $[-1,2)\times[-K,2K)$,

$$
\begin{aligned}
\|Hg\|_{M^{p,q}_w} \asymp \|V_\varphi Hg\|_{L^{p,q}} &= \Big\|\Big(\sum_{n\in\mathbb{Z}}\int_{\frac{n}{K}}^{\frac{n+1}{K}} |V_\varphi Hg(u,\nu)w_1(u)|^p\Big)^{\frac1p}\Big\|_{L^q_{w_2}}\\
&= \Big\|\Big(\sum_{j'=0}^{L}\int_{\frac{k_{j'}}{K}}^{\frac{(k_{j'}+1)}{K}}\int_{\frac{l_{j'}K}{L}}^{\frac{(l_{j'}+1)}{L}}\int \big|V_{\varphi\otimes\varphi}\widetilde{\eta}(u,\xi,\nu,x)w_1(x)\big|^p\,d\xi\,dx\,du\Big)^{\frac1p}\Big\|_{L^q_{w_2}}\\
&= \Big\|\Big(\int_0^1\!\!\int_0^K\!\!\int \big|V_{\varphi\otimes\varphi}\widetilde{\eta}(u,\xi,\nu,x)w_1(x)\big|^p\,d\xi\,dx\,du\Big)^{\frac1p}\Big\|_{L^q_{w_2}}\\
&\asymp \Big\|\Big(\int_0^1\!\!\int_0^K\!\!\int \big|V_\varrho\widetilde{\eta}(u,\xi,\nu,x)w_1(x)\big|^p\,d\xi\,dx\,du\Big)^{\frac1p}\Big\|_{L^q_{w_2}}\\
&\geq \Big\|\Big(\int_0^1\!\!\int_0^K\!\!\int \big|\chi_{[0,1)}(u)\chi_{[0,K)}(\xi)V_\varrho\widetilde{\eta}(u,\xi,\nu,x)w_1(x)\big|^p\,d\xi\,dx\,du\Big)^{\frac1p}\Big\|_{L^q(\nu)}\\
&\geq \Big\|\Big(\int_0^1\!\!\int_0^K\!\!\int \big|\chi_{[0,1)}(u)\chi_{[0,K)}(\xi)\mathcal{F}^s\widetilde{\eta}(\nu,x)w_1(x)\big|^p\,d\xi\,dx\,du\Big)^{\frac1p}\Big\|_{L^q(\nu)}\\
&= \Big\|\Big(K\int_0^K \big|\mathcal{F}^s\widetilde{\eta}(\nu,x)w_1(x)\big|^p\,dx\Big)^{\frac1p}\Big\|_{L^q_{w_2}} = \Big\|\Big(K\int_0^K \big|\widetilde{\sigma}(x,\nu)w_1(x)\big|^p\,dx\Big)^{\frac1p}\Big\|_{L^q_{w_2}}\\
&= \|\widetilde{\sigma}\|_{L^{p,q}_w} \asymp \|\sigma\|_{L^{p,q}_w}\,.
\end{aligned}
\tag{33}
$$

To obtain (33), we apply a mixed $L^p$-space version of Young's inequality for convolutions, namely, we use that for $\widetilde{\varrho}(x,\xi)=e^{2\pi ix\xi}\varrho(x,\xi)\in L^1(\mathbb{R})$, we have $\widetilde{\sigma}(x,\xi)=\widehat{\widetilde{\varrho}}*\sigma$ and $\sigma(x,\xi)=\widehat{\widehat{\widetilde{\varrho}}}*\widetilde{\sigma}$ (see Theorem 11.1, [Grö01]).

## 4.2 Outline of the proof of Theorem 4.7

We omit detailed computations as they would closely resemble computations carried out in [KP06, Pfa08b, PW06b, PW06a]. For the interested reader, we suggest to use [PW06a] as a companion when filling in detail.

We shall show that for a measurable subset $M$ with $\mathrm{vol}^-(M)>1$, the operator class $OPW^{p,q}_w(M)$ is not identifiable. In detail, we shall show that for *every* $g\in M^{\infty,\infty}(\mathbb{R})$, the operator

$$\Phi_g : OPW^{p,q}_w(M) \longrightarrow M^{p,q}_w(\mathbb{R}),\ H\mapsto Hg\,,$$

is *not* bounded below, that is, there exists *no* $c>0$ for which we have $\|Hg\|_{M^{p,q}_w}\geq c\|\sigma_H\|_{L^{p,q}_w}$ for all $H\in OPW^{p,q}_w(M)$.

$$\begin{tikzcd}
OPW^{p,q}_w(M) \arrow[r, "\Phi_g"] & M^{p,q}_w(\mathbb{R}) \arrow[d, "C_{(\mathfrak{g},\,u\mathbb{Z}^{2d})}"] \\
l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2) \arrow[u, "D_{\{P_j\}}"] \arrow[r, "M"] & l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2)
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum_j c_j P_j \arrow[r, "\Phi_g"] & \sum_j c_j P_j g \arrow[d, "C_{(\mathfrak{g},\,u\mathbb{Z}^{2d})}"] \\
\{c_j\}_j \arrow[u, "D_{\{P_j\}}"] \arrow[r, "M"] & \{\sum c_j \langle P_j g, \pi(uj')\mathfrak{g}\}_{j'}
\end{tikzcd}$$

Figure 3: Sketch of the proof of Theorem 4.7. We choose a structured operator family $\{P_j\} \subseteq OPW^{p,q}_w(M)$ so that the corresponding synthesis map $D_{\{P_j\}} : \{c_j\} \longrightarrow \sum c_j P_j$ has a bounded left inverse. Note that $C_{(\mathfrak{g},\,u\mathbb{Z}^{2d})}$ has a bounded left inverse for $u < 1$ as well. Theorem 4.13 shows that for any $g \in M^\infty(\mathbb{R})$, the composition $M = C_{(\mathfrak{g},\,u\mathbb{Z}^{2d})} \circ \phi_g \circ D_{\{P_j\}}$ has *no* bounded left inverses. This implies that $\phi_g : OPW^{p,q}_w(M) \longrightarrow M^{p,q}_w(\mathbb{R})$ also has *no* bounded left inverses.

To this end, choose $K,\, L \in \mathbb{N}$ and $V_M = \bigcup_{j=0}^{L-1} \Big(R_{K,L} + (\frac{m_j}{K}, \frac{n_j K}{L})\Big)$, $\quad m_j, n_j \in \mathbb{Z}$, where $R_{K,L} = [0, \frac{1}{K}) \times [0, \frac{K}{L})$ and where $(m_j, n_j) \neq (m_{j'}, n_{j'})$ if $j \neq j'$, such that $V_M \subseteq M$ and $\mathrm{vol}(V_M) > 1$. It is sufficient to show that $OPW^{p,q}_w(V_M)$ is not identifiable as $OPW^{p,q}_w(V_M) \subseteq OPW^{p,q}_w(M)$.

The proof of Theorem 4.7 is also sketched in Figure 3. The proof is based on extensions to results from [Pfa08b, PW06a] which are stated below and which concern the construction of the operator family $\{P_j\}$ in Figure 3.

**Lemma 4.11** *Let $P : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. For $p,\, r \in \mathbb{R}$ and $w,\, \xi \in \mathbb{R}$, define $\widetilde{P} = M_w T_{p-r} P T_r M_{\xi - w} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. Then $\eta_{\widetilde{P}} = e^{2\pi i r \xi} M_{(w,r)}\, T_{(p,\xi)}\, \eta_P$.*

**Lemma 4.12** *Fix $u > 1$ with $1 < u^4 < \frac{J}{L}$ and $0 < \epsilon < 1$. Choose $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$ with values in $[0,1]$,*

$$\eta_1(t) = \begin{cases} 1 & \text{for } t \in [\frac{u-1}{2uK}, \frac{u+1}{2uK}) \\ 0 & \text{for } t \notin [0, \frac{1}{K}) \end{cases} \quad \text{and} \quad \eta_2(\nu) = \begin{cases} 1 & \text{for } \nu \in [\frac{(u-1)K}{2uL}, \frac{(u+1)K}{2uL}) \\ 0 & \text{for } \nu \notin [0, \frac{K}{L}) \end{cases},$$

*and $|\mathcal{F}\eta_1(\xi)| \leq C e^{-\gamma|\xi|^{1-\epsilon}}$, $|\mathcal{F}^{-1}\eta_2(x)| \leq C e^{-\gamma|x|^{1-\epsilon}}$ . Let $\eta_P = \eta_1 \otimes \eta_2$. Then $\operatorname{supp} \eta_P \subseteq [0, \frac{1}{K}) \times [0, \frac{K}{L}) = R_{K,L}$ and the operator $P \in OPW^{1,1}(R_{K,L})$ has the following properties:*
*a) The operator family*

$$\Big\{ M_{uKk}\, T_{\frac{1}{K}m - \frac{uL}{K}l}\, P\, T_{\frac{uL}{K}l}\, M_{\frac{K}{L}n - uKk} \Big\}_{k,l,m,n \in \mathbb{Z}}$$

*is an $l^{p,q}_{\widetilde{w}}$–Riesz basis sequence for $OPW^{p,q}_w(\mathbb{R}^2)$.*
*b) $P \in OPW^{1,1}(R_{K,L})$ and there exist functions $d_1, d_2 : \mathbb{R} \to \mathbb{R}^+_0$ with*

$$|Pf(x)| \leq \|f\|_{M^{\infty,\infty}}\, d_1(x) \quad \text{and} \quad |\widehat{Pf}(\xi)| \leq \|f\|_{M^{\infty,\infty}}\, d_2(\xi), \quad f \in M^{\infty,\infty}(\mathbb{R}),$$

*and $d_1(x) \leq \widetilde{C} e^{-\widetilde{\gamma}|x|^{1-\epsilon}}$, $d_2(\xi) \leq \widetilde{C} e^{-\widetilde{\gamma}|x|^{1-\epsilon}}$ for some $\widetilde{C}, \widetilde{\gamma} > 0$.*

*Proof.* The existence of $\eta_1, \eta_2$ satisfying the hypotheses stated above is established through mollifying characteristic functions. In fact, using constructions of Gevrey class functions, it has been shown

that for $\epsilon, \delta > 0$, there exists $\varphi : \mathbb{R} \to \mathbb{R}_0^+$ and $C, \gamma > 0$ with $\operatorname{supp}\varphi \subseteq [-\delta, \delta]$, $\int \varphi = 1$, $\widehat{\varphi}(\xi) \leq Ce^{-\gamma|\xi|^{1-\epsilon}}$[Hör03, Hör05, DH98]. Note that the restriction to $w$ subexponential in Theorem 4.7 is a consequence to the fact that there exist no compactly supported functions whose Fourier transforms decay exponentially (see references in [GP08]).

*a)* Due to Lemma 4.11,

$$\left\{ M_{(uKk, \frac{uL}{K}l)}\, T_{(\frac{1}{K}m, \frac{K}{L}n)}\ \eta_P \right\}_{k,l,m,n \in \mathbb{Z}}$$

being an $l^{p,q}_{\widetilde{w}}$–Riesz basis for its closed linear span in $M^{(1,1),(p,q)}_{1\otimes w}(\mathbb{R}^2)$ implies that

$$\left\{ M_{uKk}\, T_{\frac{1}{K}m - \frac{uL}{K}l}\, P\, T_{\frac{uL}{K}l}\, M_{\frac{K}{L}n - uKk} \right\}_{k,l,m,n \in \mathbb{Z}}$$

is an $l^{p,q}_{\widetilde{w}}$-Riesz basis for its closed linear span in $OPW^{p,q}_w(\mathbb{R}^2)$.

*b)* As shown in the proof of Lemma 3.4 in [KP06], we have $|Pf(x)| \leq |\widehat{\eta}_2(-x)|\ \ \|f\|_{M^{\infty,\infty}} \|\eta_1\|_{M^{1,1}}$, so we can choose $d_1(x) = |\widehat{\eta}_2(-x)|\, \|\eta_1\|_{M^{1,1}}$.

Similarly, we can compute $|\widehat{Pf}(\xi)| \leq \|f\|_{M^{\infty,\infty}} \|\eta_2(\xi - \cdot)\widehat{\eta}_1(\cdot)\|_{M^{1,1}}$. We claim that $d_2(\xi) = \|\eta_2(\xi - \cdot)\widehat{\eta}_1(\cdot)\|_{M^{1,1}}$ has the postulated subexponential decay. Recall that for $g$ supported on $[a,b]$, we have $\|g\|_{M^{1,1}} \leq c\ \|\widehat{g}\|_{L^1}$ where $c$ depends only on the support size $b - a$ (see Lemma 2.3 and [Oko09]). As $\eta_2(\xi - \cdot)\widehat{\eta}_1(\cdot)$ is compactly supported with uniform support size, we can compute

$$\begin{aligned} d_2(\xi) &= \|\eta_2(\xi - \cdot)\widehat{\eta}_1(\cdot)\|_{M^{1,1}} \leq c\ \|\mathcal{F}^{-1}\big(\eta_2(\xi - \cdot)\widehat{\eta}_1(\cdot)\big)\|_{L^1} \\ &= c\ \int \Big| \int \eta_2(\xi - \nu)\widehat{\eta}_1(\nu) e^{2\pi i x\nu}\, d\nu \Big|\, dx = c\ \int \Big| V_{\widetilde{\eta}_2}\widehat{\eta}_1(\xi, -x) \Big|\, dx\,, \end{aligned} \tag{34}$$

where $\widetilde{\eta}_2(\xi) = \eta_2(-\xi)$. As the compact support of $\eta_1, \eta_2$ together with $|\mathcal{F}\eta_1(\xi)| \leq Ce^{-\gamma|\xi|^{1-\epsilon}}$, $|\mathcal{F}^{-1}\eta_2(x)| \leq Ce^{-\gamma|x|^{1-\epsilon}}$ imply that $\widehat{\eta}_1, \widetilde{\eta}_2$ are in the Gelfand–Shilov class $\mathcal{S}^{1-\epsilon}_{1-\epsilon}$ [GS68], we apply Proposition 3.12 in [GZ04] to conclude that $V_{\widetilde{\eta}_2}\widehat{\eta}_1 \in \mathcal{S}^{1-\epsilon}_{1-\epsilon}$, that is,

$$d_2(\xi) \ \leq\ c\ \int \widetilde{C}^{-1} e^{-\widetilde{\gamma}\|(x,y)\|_\infty^{1-\epsilon}}\, dx \ \leq\ \widetilde{\widetilde{C}}\, e^{-\widetilde{\gamma}|\xi|^{1-\epsilon}}\,.$$

□

Theorem 4.13 extends the main result in [Pfa08b] to weighted and mixed $l^p$ spaces with subexponential weights. Both results generalize to infinite dimensions the fact that $m \times n$ matrices with $m < n$ have a non–trivial kernel and, therefore, are not bounded below as operators acting on $\mathbb{C}^n$.

**Theorem 4.13** *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $w_1, w_2$ subexponential on $\mathbb{Z}^{2d}$, that is, for some $C, \gamma > 0$, $0 < \beta < 1$, we have*

$$C^{-1}\, e^{-\gamma\|n\|_\infty^\beta} \leq w_1(n), w_2(n) \leq C\, e^{\gamma\|n\|_\infty^\beta}\,. \tag{35}$$

*If for $M = (m_{j'j}) : l^{p_1,q_1}_{w_1}(\mathbb{Z}^{2d}) \to l^{p_2,q_2}_{w_2}(\mathbb{Z}^{2d})$, exists $u > 1$, $K_0 > 0$, and*

$$\rho : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+ \quad \textit{with} \quad \rho \leq \widetilde{C}\, e^{-\widetilde{\gamma}\|n\|_\infty^{\widetilde{\beta}}}, \quad \widetilde{\beta} > \beta,$$

*with*

$$|m_{j'j}| \leq \rho(u\|j'\|_\infty - \|j\|_\infty), \quad u\|j'\|_\infty - \|j\|_\infty > K_0,$$

*then $M$ has no bounded left inverses.*

*Proof.* Let $v(n) = C\, e^{\gamma\|n\|_\infty^\beta}$. Note that $l^{p_1,q_1}_{w_1}(\mathbb{Z}^{2d})$ embeds continuously in $l^{\infty,\infty}_{1/v}(\mathbb{Z}^{2d}) = l^{\infty}_{1/v}(\mathbb{Z}^{2d})$ and $l^{1,1}_{v}(\mathbb{Z}^{2d}) = l^{1}_{v}(\mathbb{Z}^{2d})$ embeds continuously in $l^{p_2,q_2}_{w_2}(\mathbb{Z}^{2d})$. Hence, it suffices to show that for all $\epsilon>0$ exists $x\in l^\infty_{1/v}(\mathbb{Z}^{2d})$ with $\|x\|_{l^\infty_{1/v}}=1$ and $\|Mx\|_{l^1_v}\leq \epsilon$. For notational simplicity, we replace $2d$ by $D$ in the following.

First, observe that

$$A_{K_1} = e^{\gamma K_1^\beta} \sum_{K\geq K_1} K^{D-1} e^{\gamma K^\beta} \sum_{k\geq K} k^{D-1}\, e^{-\widetilde{\gamma} k^{\widetilde{\beta}}} \to 0 \ \text{ as } K_1\to\infty. \tag{36}$$

Applying the integral criterion for sums, this would follow from

$$e^{\gamma K_1^\beta} \int_{K_1}^\infty x^{D-1} e^{\gamma x^\beta} \int_x^\infty y^{D-1}\, e^{-\widetilde{\gamma} y^{\widetilde{\beta}}}\, dy\, dx \to 0 \ \text{ as } K_1\to\infty. \tag{37}$$

For large $x$, a substitution yields

$$\begin{aligned}
\int_x^\infty y^{D-1}\, e^{-\widetilde{\gamma} y^{\widetilde{\beta}}}\, dy &= \frac{1}{\widetilde{\beta}\widetilde{\gamma}} \int_{\widetilde{\gamma} x^{\widetilde{\beta}}}^\infty \big(\tfrac{t}{\widetilde{\gamma}}\big)^{\frac{1}{\widetilde{\beta}}(D-1)} \big(\tfrac{t}{\widetilde{\gamma}}\big)^{\frac{1-\widetilde{\beta}}{\widetilde{\beta}}(D-1)} e^{-t}\, dt = \frac{1}{\widetilde{\beta}\widetilde{\gamma}} \widetilde{\gamma}^{2(D-1)(1-\frac{1}{\widetilde{\beta}})} \int_{\widetilde{\gamma} x^{\widetilde{\beta}}}^\infty t^{2\frac{D-1}{\widetilde{\beta}}-D+2-1} e^{-t} dt \\
&= \frac{1}{\widetilde{\beta}\widetilde{\gamma}} \widetilde{\gamma}^{2(D-1)(1-\frac{1}{\widetilde{\beta}})}\, \Gamma\big(2\tfrac{D-1}{\widetilde{\beta}}-D+2, \widetilde{\gamma} x^{\widetilde{\beta}}\big) \leq \frac{2}{\widetilde{\beta}\widetilde{\gamma}} \widetilde{\gamma}^{2(D-1)(1-\frac{1}{\widetilde{\beta}})} \big(\widetilde{\gamma} x^{\widetilde{\beta}}\big)^{2\frac{D-1}{\widetilde{\beta}}-D+1}\, e^{-\widetilde{\gamma} x^{\widetilde{\beta}}} \\
&\leq \frac{2}{\widetilde{\beta}} \widetilde{\gamma}^{D-2}\, x^{2D-1-\widetilde{\beta}(D+1)}\, e^{-\widetilde{\gamma} x^{\widetilde{\beta}}},
\end{aligned}$$

where $\Gamma$ denotes the upper incomplete Gamma function, and we used $\frac{\Gamma(s,y)}{y^{s-1}e^{-y}}\to 1$ as $y\to\infty$ [AS92].

The fact that $\widetilde{\beta}>\beta$ allows us to estimate for large $x$

$$e^{\gamma x^\beta} e^{-\widetilde{\gamma} x^{\widetilde{\beta}}} = e^{-\widetilde{\gamma} x^{\widetilde{\beta}}(1-\frac{\gamma}{\widetilde{\gamma}} x^{\beta-\widetilde{\beta}})} \leq e^{-\frac{1}{2}\widetilde{\gamma} x^{\widetilde{\beta}}}.$$

Hence, we can repeat the arguments above to the outer integral and the product with $e^{\gamma K_1^\beta}$ in (37) to obtain (37), and, hence, (36) holds.

To continue our proof, we fix $\epsilon>0$ and note that (36) provides us with a $K_1>K_0$ satisfying $A_{K_1}\leq (2^{2D}D^2\,\widetilde{C}\,C^2\, e^{\gamma N^\beta})^{-1}\,\epsilon$.

As in [Pfa08b], set $N=\left\lceil \frac{\lambda(K_1+1)}{\lambda-1}\right\rceil$ and $\widetilde{N}=\lceil\frac{N}{\lambda}\rceil+K_1$. Then $\frac{\lambda(K_1+1)}{\lambda-1}\leq N\leq \frac{\lambda(K_1+2)}{\lambda-1}$ implies $\lambda N\geq \lambda K_1+\lambda+N$ and $N\geq K_1+\frac{N}{\lambda}+1>K_1+\lceil\frac{N}{\lambda}\rceil=\widetilde{N}$. Hence, $(2\widetilde{N}+1)^D<(2N+1)^D$ so that the matrix $\widetilde{M}=(m_{j'j})_{\|j'\|_\infty\leq\widetilde{N},\|j\|\leq N}:\mathbb{C}^{(2N+1)^D}\longrightarrow \mathbb{C}^{(2\widetilde{N}+1)^D}$ has a nontrivial kernel. We now choose $x\in l^\infty_{1/v}(\mathbb{Z}^D)$ with $\|x\|_{l^\infty_{1/v}}=1$, $x_j=0$ if $\|j\|_\infty>N$, and and $\widetilde{M}\widetilde{x}=0$ where $\widetilde{x}$ is $x$ restricted to the set $\{j:\ \|j\|_\infty\leq N\}$.

By construction we have $(Mx)_{j'}=0$ for $\|j'\|_\infty\leq\widetilde{N}$. To estimate $(Mx)_{j'}$ for $\|j'\|_\infty>\widetilde{N}$, we fix $K>K_1$ and one of the $2D\big(2(\lceil\frac{N}{\lambda}\rceil+K)\big)^{D-1}$ indices $j'\in\mathbb{Z}^D$ with $\|j'\|_\infty=\lceil\frac{N}{\lambda}\rceil+K$. We have $\|\lambda j'\|_\infty\geq N+K\lambda$ and $\lambda\|j'\|_\infty-\|j\|_\infty\geq K\lambda\geq K$ for all $j\in\mathbb{Z}^D$ with $\|j\|_\infty\leq N$. Therefore, using Hölder's inequality for weighted $l^p$-spaces, we obtain

$$\begin{aligned}
|(Mx)_{j'}| &= \Big|\sum_{\|j\|_\infty\leq N} m_{j'j}x_j\Big| \leq \|x\|_{l^\infty_{1/v}} \sum_{\|j\|_\infty\leq N} v(j)\ |m_{j'j}| \\
&\leq C\,e^{\gamma N^\beta} \sum_{\|j\|_\infty\leq N} \rho(\lambda\|j'\|_\infty-\|j\|_\infty) \leq C\,e^{\gamma N^\beta} \sum_{\|j\|_\infty\geq K} \rho(\|j\|_\infty) = 2^D D\,C\,e^{\gamma N^\beta} \sum_{k\geq K} k^{D-1}\rho(k).
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\|Mx\|_{l^1_v} &= \sum_{j'\in\mathbb{Z}^D} v(j')\,|(Mx)_{j'}| = \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda}\rceil + K_1} v(j')\,|(Mx)_{j'}| \\
&\leq 2^D D\, C\, e^{\gamma N^\beta} \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda}\rceil + K_1} v(j') \sum_{k\geq \|j'\|_\infty} k^{D-1}\rho(k) \\
&\leq 2^D D\, C\, e^{\gamma N^\beta} \sum_{K \geq \lceil \frac{N}{\lambda}\rceil + K_1} 2D(2K)^{D-1} C\, e^{\gamma K^\beta} \sum_{k\geq K} k^{D-1}\,\rho(k) \\
&\leq 2^{2D} D^2\, \widetilde{C}\, C^2\, e^{\gamma N^\beta} \sum_{K \geq \lceil \frac{N}{\lambda}\rceil + K_1} K^{D-1} e^{\gamma K^\beta} \sum_{k\geq K} k^{D-1}\, e^{-\widetilde{\gamma} k^{\widetilde{\beta}}} \leq \epsilon\,.
\end{aligned}$$

□

Combining the results above, we can now proceed to prove Theorem 4.7.

*Proof of Theorem 4.7.* As $w$ is subexponential, there exists $C, \gamma, \epsilon > 0$ with $|w(x,\xi)| \leq C\, e^{\gamma \|(x,\xi)\|_\infty^{1-2\epsilon}}$. For this $\epsilon > 0$ choose $u$, $\eta_1$, $\eta_2$, $P$, $d_1$, and $d_2$ as in Lemma 4.12.

Define the synthesis operator $E : l^{p,q}_w(\mathbb{Z}^2) \to OPW^{p,q}_w(V_M) \subseteq OPW^{p,q}_w(M)$ as follows. For $\sigma = \{\sigma_{k,p}\} \in l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2)$ write $\sigma_{k,p} = \sigma_{k,lJ+j}$ for $l \in \mathbb{Z}$ and $0 \leq j < J$ and define

$$E(\sigma) = \sum_{k,l\in Z} \sum_{j=0}^{J-1} \sigma_{k,lJ+j}\; M_{uKk}\, T_{\frac{1}{K}k_j + \frac{uL}{K}l}\, P\, T_{-\frac{uL}{K}l}\, M_{\frac{K}{L}p_j - uKk} \tag{38}$$

with convergence in case $p, q \neq \infty$ and weak-$*$ convergence else. Since

$$\left\{ M_{uKk}\, T_{\frac{1}{K}m - \frac{uL}{K}l}\, P\, T_{\frac{uL}{K}l}\, M_{\frac{K}{L}n - uKk} \right\}_{k,l,m,n\in\mathbb{Z}}$$

is an $l^{p,q}_{\widetilde{w}}$–Riesz basis for its closed linear span in $OPW^{p,q}_w(\mathbb{R}^2)$, so is its subset

$$\left\{ M_{uKk}\, T_{\frac{1}{K}k_j + \frac{uL}{K}l}\, P\, T_{-\frac{uL}{K}l}\, M_{\frac{K}{L}p_j - uKk} \right\}_{k,l\in\mathbb{Z}, 0\leq j<J}$$

and $E$ is bounded and bounded below .

By Theorem 2.5, the Gabor system $(\mathfrak{g}, a'\mathbb{Z}\times b'\mathbb{Z}) = \{M_{ka'}T_{lb'}\mathfrak{g}\}$ is an $l^{p,q}_{\widetilde{w}}$–frame for any $a', b' > 0$ with $a'b' < 1$, and we conclude that the analysis map given by

$$C_{\mathfrak{g}} : M^{p,q}_w(\mathbb{R}) \to l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2), \quad f \mapsto \left\{ \langle f, M_{u^2K\,k} T_{\frac{u^2L}{KJ}\,l}\, \mathfrak{g} \rangle \right\}_{k,l} \tag{39}$$

is bounded and bounded below since $u^2K\frac{u^2L}{KJ} = u^4\frac{L}{J} < 1$.

For simplicity of notation, set $\alpha = K$ and $\beta = \frac{L}{KJ}$. Fix $f \in M^{\infty,\infty}(\mathbb{R})$ and consider the composition

$$\begin{array}{ccccccc}
l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2) & \xrightarrow{E} & OPW^{p,q}_w(M) & \xrightarrow{\Phi_{\mathfrak{g}}} & M^{p,q}_w(\mathbb{R}) & \xrightarrow{C_{\mathfrak{g}}} & l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2) \\
\sigma & \mapsto & E\sigma & \mapsto & E\sigma\, g & \mapsto & \{\, \langle\, E\sigma\, g,\ M_{u^2\alpha k'} T_{u^2\beta l'}\, \mathfrak{g}\, \rangle\, \}_{k',l'}\,.
\end{array}$$

It is easily computed that the operator $C_{\mathfrak{g}} \circ \Phi_f \circ E$ is represented — with respect to the canonical basis $\{\delta(\cdot - n)\}_n$ of $l^{p,q}_{\widetilde{w}}(\mathbb{Z}^2)$ — by the bi-infinite matrix

$$\mathcal{M} = \Big( m_{k',l',\,k\,,lJ+j} \Big) = \Big( \langle\; M_{u\alpha k}\, T_{\frac{k_j}{\alpha} + u\beta lJ}\, P\, T_{-u\beta lJ}\, M_{\frac{p_j}{\beta J} - u\alpha k}\, f\,,\ M_{u^2\alpha k'} T_{u^2\beta l'}\, \mathfrak{g}\, \rangle \Big)\,.$$

Setting

$$\widetilde{d}_1 = \max_{j=0,\dots,J-1} T_{\frac{k_j}{\alpha}-\lambda\beta j} d_1 \,,$$

we observe

$$\begin{aligned}
|m_{k',l',k,lJ+j}| &\leq \Big\langle T_{u\beta(lJ+j)}\big(T_{\frac{k_j}{\alpha}-u\beta j}\Big| P\,T_{-u\beta lJ}\,M_{\frac{p_j}{\beta J}-u\alpha k}\,f\Big|\big)\,, T_{u^2\beta l'}\mathfrak{g}\Big\rangle \\
&\leq \|f\|_{M^{\infty,\infty}}\,\big\langle T_{u\beta(lJ+j)} T_{\frac{k_j}{\alpha}-u\beta j} d_1\,,\ T_{u^2\beta l'}\mathfrak{g}\big\rangle \\
&\leq \|f\|_{M^{\infty,\infty}}\,(\widetilde{d}_1 * \mathfrak{g})\,(u\beta(ul' - (lJ+j)))\,,
\end{aligned}$$

and

$$\begin{aligned}
|m_{k',l',k,lJ+j}| &= \Big|\Big\langle T_{u\alpha k}\,M_{-\frac{k_j}{\alpha}-u\beta lJ}\,\big(P\,T_{-u\beta lJ}\,M_{\frac{p_j}{\beta J}-u\alpha k}\,f\big)\widehat{\ }, T_{u^2\alpha k'}M_{-u^2\beta l'}\mathfrak{g}\Big\rangle\Big| \\
&\leq \Big\langle T_{u\alpha k}\Big|\big(P\,T_{-u\beta lJ}\,M_{\frac{p_j}{\beta J}-u\alpha k}\,f\big)\widehat{\ }\Big|\,,\ T_{u^2\alpha k'}\mathfrak{g}\Big\rangle \\
&\leq \|f\|_{M^{\infty,\infty}}\,(d_2 * \mathfrak{g})(u\alpha(uk'-k)).
\end{aligned}$$

Observing that for appropriate $\widetilde{C}, \widetilde{\gamma}$, we have $\widetilde{d}_1 * \mathfrak{g}(x) d_2 * \mathfrak{g}(\xi) \leq \widetilde{C}\, e^{-\widetilde{\gamma}\|(x,\xi)\|_\infty^{1-\epsilon}}$ and $1-2\epsilon < 1-\epsilon$, allows us to apply Theorem 4.13 to $\mathcal{M}$. This completes the proof. □

# 5 Outlook

Recent results in operator identification with relevance to operator sampling include, for example, the identification of Multiple Input Multiple Output (MIMO) channels [Pfa08a] and an extension of operator sampling to irregular sampling sets [HP09]. Some more fundamental questions concerning sampling and identification of operator Paley–Wiener spaces are still open. In the following we describe two such questions.

**Unbounded spreading domains with small Lebesgue measure.**

The extension of Theorem 4.6 to $OPW_w^{p,q}(M)$ with $M$ unbounded but with Lebesgue measure less than one remains open. The following observations encourage tackling this question:

1. Multiplication operators with not necessarily bandlimited symbol in $L^2(\mathbb{R}^2)$ are clearly identifiable with identifier $g = \chi_{\mathbb{R}} \in M^{\infty,\infty}(\mathbb{R})$. Note that the characteristic function $\chi_{\mathbb{R}}$ is the weak-$*$ limit of $T\sum_{n\in\mathbb{Z}} \delta_{nT}$ as $T \to 0$. Hence, the space $OPW^{\infty,\infty}\big(\{0\}\times\widehat{\mathbb{R}}\big)$ is identifiable.

2. Time–invariant operators with not necessarily compactly supported $L^2(\mathbb{R}^2)$ impulse response are identifiable with identifier $\delta$ which is the weak-$*$ limit of $\sum_{n\in\mathbb{Z}} \delta_{nT}$ as $T \to \infty$. Consequently, the space $OPW^{\infty,\infty}\big(\mathbb{R}\times\{0\}\big)$ is identifiable.

3. In [KP06] it is shown that $OPW(M)$ is identifiable if $M$ is a possibly unbounded fundamental domain a lattice $\Lambda$ in $\mathbb{R}^2$ with $\Lambda$ having density less than or equal to one. This result covers, for example, $OPW\big(\{(t,\nu)\ :\ t{\geq}-1,\ \nu{<}1,\ 2^{-(t+1)}{\leq}\nu{\leq}2^{-t}\}\big)$ as the unbounded set $\big\{(t,\nu)\ :\ t{\geq}-1,\ \nu{<}1,\ 2^{-(t+1)}{\leq}\nu{\leq}2^{-t}\big\}$ is a fundamental domain of $\mathbb{Z}^2$.

The natural approach to construct identifiers for $OPW(A)$ as weak-$*$ limit of identifiers $g_N$ for $OPW(A\cap[-N,N]\times[-N,N])$ is difficult as the constants implied by $\asymp$ in (26) depend in a non-trivial matter on $g_N = \sum_n c_{n,N}\delta_{x_{n,N}}$, $N\in\mathbb{N}$, if the sequences $\{c_{n,N}\}$ are not constant.

**Generalizations to higher dimensions.**

As mentioned in Section 4, our proof of Theorem 4.6 hinges on the existence of identifiers in an analogous setup where the locally compact Abelian group $\mathbb{R}$ is replaced by an appropriate finite cyclic group of prime order $\mathbb{Z}_p$ [KPR08, LPW05]. In fact, generalizing Theorem 4.6, to operators acting on $L^2(\mathbb{R}^d)$ would be possible if the conclusions of Theorem 4.9 hold for sufficiently many composites $n$ taking the place of prime $p$. In fact, in [KPR08], we ask the following

**Question 5.1** *Is it true that for all $L \in \mathbb{N}$ exists $c \in \mathbb{C}^L$ so that the vectors $\pi(k,\ell)c$, $k,\ell \in 0,\ldots,L-1$, defined by $(\pi(k,\ell)c)_j = c_{j-k}\, e^{2\pi i \frac{j\ell}{L}}$, $k,\ell = 0,\ldots,L-1$, are in general linear position.*

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